

More on Homological Supersymmetric Quantum Mechanics

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In this work, we first solve complex Morse flow equations for the simplest case of a bosonic harmonic oscillator to discuss a localization problem in the context of the Picard-Lefschetz theory. We briefly touch on the complex phases associated with downward flow lines for supersymmetric quantum mechanics on algebraic geometric grounds and report that phases of non-BPS exact saddle solutions can be accessed through the cohomology of WKB 1-form of the underlying singular spectral curve subject to necessary cohomological corrections for non-zero genus. Motivated by the Picard-Lefschetz theory, we write down a general formula for the index of $\mathcal{N} = 4$ quantum mechanics with background R -symmetry gauge fields. We conjecture that certain symmetries of the refined Witten index and singularities of the moduli space may be used to determine the correct intersection coefficients. The R -anomaly removal along relative homology cycles also called “Lefschetz thimbles” is investigated. We show that the Fayet-Iliopoulos parameters appear in the intersection coefficients for the relative homology of the quiver quantum mechanics resulting from dimensional reduction of $2d$ $\mathcal{N} = (2, 2)$ gauge theory on a circle and explicitly calculate integrals along the Lefschetz thimbles in $\mathcal{N} = 4$ $\mathbb{P}_{k-1}(\mathbb{C})$ model. In the case of quivers with higher-rank gauge groups, several examples of index computations using Picard-Lefschetz decomposition are presented. The Stokes jumping of coefficients and its relation to wall-crossing phenomena is briefly discussed. An implication of the Lefschetz thimbles in constructing knots from quiver quantum mechanics is indicated.

1. INTRODUCTION AND SUMMARY

Supersymmetric quantum mechanics is a very fruitful tool for studying mathematical properties of manifolds and algebraic curves. It also provides tractable prototypes to discuss (algebraic) topological invariants that arise from physical systems described by vacuum (e.g. zero-energy) states of generic supersymmetric quantum field theories. Therefore, at least from a topological standpoint, supersymmetric quantum mechanics boils down to calculating the ground states and their proper counting based on their parity under the action of $(-1)^F$ operator where F is the fermion number operator. The result of this counting, considering the parity for every solution, leads to the most basic topological invariant protected by supersymmetry, namely, Witten index [1]:

$$\mathcal{I} = \text{tr}(-1)^F. \quad (1)$$

\mathcal{I} also gives the Euler characteristic for the target manifold of the underlying field theory.

The study of (1) led to a seminal work [2] where the de Rham cohomology of compact finite-dimensional manifolds was computed using a deep relation between Morse inequalities and supersymmetric quantum mechanics. Two years later, Atiyah and Bott [3] showed that de Rham version of the localization theorems in the context of equivariant cohomology are very closely related to Witten’s result and gave a generalized exact stationary phase formula. A Morse theoretic study of Yang-Mills equations over Riemann surfaces (or algebraic curves) M was done in [4] that proved useful in deriving the cohomology of the moduli spaces of stable algebraic vector bundles over M . These efforts paved the way for applying Morse theory to quantum field theories as well.

The Picard-Lefschetz theory is a more recent attempt to deliver a crucial understanding of complexified path

integrals and index technology by studying the topology of path space via the critical points of the holomorphized action. In this regard, a possible use of this theory in calculating the Witten index of $2d$ gauge theories was first addressed in $2d$ supersymmetric Landau-Ginzburg models in [5]. Not long ago, Witten introduced this theory into an analytically continued version of Chern-Simons path integral with Wilson loop operator insertions in an endeavor to recreate Jones polynomial of knots [6]. Another application was given in [7] where it was shown that the complexified Liouville theory produces correct DOZZ formula only when its path integral is evaluated on integration cycles pieced together of flow lines attached to multivalued complex saddle solutions.

More recent applications were considered in the context of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ quantum mechanics [8, 9] as well as $\mathbb{P}_{k-1}(\mathbb{C})$ models [10] from a semi-classical point of view. Analytic continuation of path integrals have been shown to be a fundamental necessity of quantum theories to yield a correct semi-classical analysis consistent with supersymmetry [11]. What was shown to be of crucial importance, was that no matter what real theories we are considering, the axiom of holomorphization is a first step to set the stage for understanding the path integrals. Above all else lies the fact that when one applies Picard-Lefschetz theory to a holomorphized theory, we would get so much than we initially asked for: Saddle solutions that were missing in the analysis of ground state properties of physical theories, seem to crack the puzzles of semi-classics wide open. These mysterious puzzles including the vanishing of gluon condensate in $\mathcal{N} = 1$ SYM theory and the positive semi-definiteness of non-perturbative contributions to the ground state energy of the bosonized $\mathcal{N} = 1$ supersymmetric quantum mechanics were finally settled down using the aforementioned ideas combined with a key data that Picard-Lefschetz theory provides:

Hidden topological angle (HTA) [12]. Each amplitude assigned to saddle point in the stationary phase formula has to carry an extra topological phase coming from the relative homology of the cycle to which it is attached. This invariant phase along the cycle is usually a sign in supersymmetric theories. “Resurgence” kicks in when the phase changes the amount of this saddle amplitude but hopefully this does not happen here and will not concern us [13]. Furthermore, we already have a holomorphized action that contains all we need.

In supersymmetric gauge theories, localization principle provides an effective 1-loop function out of a complicated path integral that needs to be integrated on $1d$ integration cycles in a finite dimensional moduli space. The calculation of path integral is then literally boiled down to picking the right integration cycle around the poles of effective 1-loop function. Localization was formally applied to a supersymmetric quantum gauge theory in [14]. Since then, this technique has been vastly used in the literature of topological/quantum gauge theories to compute the path integrals. Here, we are interested in the index of supersymmetric quantum mechanics arising from $2d$ Witten index refined by flavor/external gauge fields, leading to a categorification of (1) often called *refined* Witten index, specifically, the $2d$ index of $\mathcal{N} = (2, 2)$ theory on a torus T^2 known as “elliptic genus” which was calculated by implementing the Jeffrey-Kirwan (JK) residue operation in [15–18] and using matrix models in [19]. Following these works, Ref. [20] computed the refined index of the $\mathcal{N} = 4$ quantum mechanics resulting from the dimensional reduction of elliptic genus on a circle.

We consider the same gauged quantum mechanical theories with four real supercharges with only finitely many vacua as in [20]. The main goal is to construct the correct integration cycles by going through a Picard-Lefschetz analysis of the theory, and discuss whether the wall-crossing phenomena are resulted from deformation of integration cycle away from the Stokes rays of the path space of the localized theory e.g., u -space, that is suitably non-compact. A more explicit answer will need a regularization term in the effective 1-loop determinant, as well as a Fayet-Iliopoulos (FI) term in the original theory. When the gauge group is Abelian or has Abelian factors, this term can be introduced, which has the form

$$S_{FI} = \zeta \int d^2\theta \text{vol} \quad (2)$$

for real ζ [21].

The organization of this paper and summary of results is in order. In section [2] we use the elementary dynamics of a bosonic harmonic oscillator by solving its Morse flow equations to pave the way for understanding the idea of localization to constant paths in the context of path integrals. We first holomorphize (complexify) the configuration space of the oscillator in question and then identify the imaginary direction with the momentum in an attempt to define the conserved Hamiltonian flow which implies the localization once the boundary conditions are

correctly set. The constant paths are downward flows attached to the critical points, which are universally referred to as *Lefschetz thimbles*, form a relative homology of the non-compact path space, study of which falls under the umbrella of complex Morse and Picard-Lefschetz theories. The Hamiltonian flow then associates a topological angle to Lefschetz thimbles that in supersymmetric quantum theories would be necessary to fix their long-standing semi-classical issues. The cohomology of singular algebraic curves corresponding to the bosonic potential of supersymmetric quantum mechanics fixed at the ground state energy level, will be considered in section [3] to provide a competing method of calculating the topological invariants of the path space. We argue that for singular algebraic curves of supersymmetric quantum mechanics with non-zero genus, apart from higher order quantum corrections, the (classical) WKB 1-form (namely, that of WKB theory at zero order expansion in g) is not enough to capture the full cohomology of underlying theory and complex phases of its saddle points hidden in the topology of thimbles, so it needs to be corrected by considering the sheaf of holomorphic 1-forms. A similar idea exploiting refined holomorphic anomaly equations around singular points of algebraic curves borrowed from topological string theory has recently been proposed to also work for generic quantum mechanical systems [22]. It would likewise be apt to understand how correction terms in complex phases discussed in this paper arise from anomaly equations.

In section [4], the refined Witten index formula of supersymmetric quantum mechanics with four real supercharges will be given by considering an integration cycle γ that is along the non-compact directions of the path space. In the vast literature of index calculations, localization is the key feature of supersymmetric theories that produces a meromorphic top-form -1-loop determinant to be integrated over this path space along a given path to be determined [14]. We point out a common feature of gauge theories that is in the presence of a group action G , the saddle points will form orbits of the group, rendering them non-isolated [6]. Thus, it is appropriate to call such points “saddle rims” as they happen to be the boundaries of the path space at infinite imaginary directions. For a $U(1)$ gauge theory, the path space is made of infinitely many copies of $\mathbb{C} \setminus \{0\} \cong S^1 \times \mathbb{R}$, where the non-compact direction corresponds to the Lie algebra of the maximal torus of $U(1)$ gauge symmetry that is the Coulomb branch of the $1d$ quiver quantum mechanics [17]. Throughout this paper, we will refer to this non-compact direction in the path space as the “wall” in which the integration cycle γ dwells. Using path homotopy constraint, we form a set of Lefschetz thimbles that flow to the degenerate saddle rims and back. The elements of this set form a basis for the relative homology describing the path space and therefore the contour will be written as $\gamma = \sum_i n_i \mathcal{J}_i$ for integer n_i . The index calculated on γ will then be interpreted as “on-the-wall” index that obviously receives only contributions from thimble integrals attached to saddle rims on the boundaries of the path space. Exactness of saddle point

approximation will come in handy to avoid difficulties of taking these integrals in general quiver theories.

Since the wall is shared by the chambers in FI parameter space (e.g. ζ -space), γ should depend on ζ . On the other hand, it is on a Stokes ray connecting distinct saddle rims so we need a general prescription to determine the correct integration cycle. This will be done in linear quiver systems of total gauge group $G = U(1)^\alpha$ that will be the subject of section [5], and closed quivers in [6]. We will observe that adding a gauge-invariant ζ -dependent regularization term will give rise to unique Stokes jumps off of γ as different chambers in ζ -space are probed in the process. Integration contour would be modified through this addition explicitly at the expense of losing non-compactness of the path-space. In all the examples studied we find that the proposed index formula exactly reproduces the results of [20].

Section [7] is intended to elaborate on the simple example of 2-node quiver theory in detail. In subsection [7.1], we solve the alternative condition on the thimbles, namely the imaginary part of effective action is constant on the flow lines attached to the same saddle rims. Subsection [7.2] shows the connection between intersection coefficients and FI parameters starting from the $2d$ theory. Being on a Stokes ray means that the effective action should satisfy a certain condition as studied in [6]. In subsection [7.3], we show that this condition imposes a constraint on the R -symmetry fugacity y that in turn forces the R -anomaly removal as is the case for the chiral theories where only one type of fermions (left or right) would couple to external gauge fields present in the original $\mathcal{N} = (2, 2)$ theory. Section [8] is devoted to structuring a paradigm in which the data concerning integration cycle on the wall tells about the value of index in all chambers as well as the jumping of Lefschetz thimbles.

Finally, we discuss a possible future direction of research in section [9] regarding knots. There, we make a connection between the HOMFLY polynomial of an unknot and refined Witten index of $\mathcal{N} = 4$ $\mathbb{P}_{k-1}(\mathbb{C})$ model and hint at possibility of projecting a mirror symmetric knot K onto the path space of some $\mathcal{N} = 4$ quiver quantum mechanics in such a way that good regions and Lefschetz thimbles are identified with crossing points and line segments between crossings of K , respectively.

2. REVIEW OF PICARD-LEFSCHETZ THEORY, AXIOM OF HOLOMORPHIZATION AND LOCALIZATION

In d dimensions, Euclidean path integral of a (non)relativistic field $\phi := \phi(\mathbf{x}, t)$ propagating from point \mathbf{x}_i to \mathbf{x}_f reads

$$Z = \int \mathcal{D}\phi e^{-S[\phi(\mathbf{x}, t)]}, \quad (3)$$

subject to the boundary conditions $\phi(\mathbf{x}_i, t_i) = \phi_i$ and $\phi(\mathbf{x}_f, t_f) = \phi_f$. Here $\mathbf{x} := \{x_i\}_{i=1}^{d-1}$. A natural Picard-

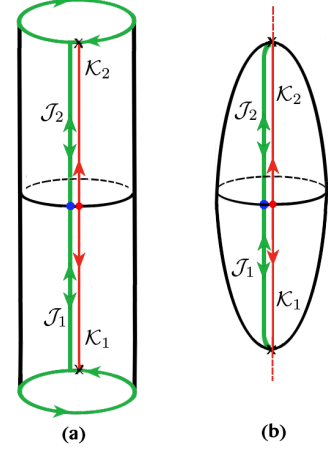


Figure 1. (a) Path space of the $U(1)$ gauge theory (b) Path space of the *partially* compactified theory by adding a gauge-invariant regularization term. The saddle rims have been reduced to fixed points of the action of $G = U(1)$ on the sphere [23]. Those are the non-degenerate isolated critical points of the regularized effective action. A basis for homology cycles \mathcal{J}_i in (a) is determined modulo path homotopy. The first ordinary homology is seen to be of rank 2 upon regularization.

Lefschetz theory treatment of path integrals requires holomorphization of fields and coordinates as $\phi \rightarrow \hat{\phi}$ and promoting the action to a *holomorphic* action $\hat{\mathbf{S}}[\hat{\phi}(z)]$ such that $\hat{S} = \int dt \hat{\mathbf{S}} \equiv h + i\theta_i$ where $h = \Re(\hat{S})$ is a Morse function (a real valued function with nondegenerate critical points, that here is chosen to be real part of a holomorphic action so should no dependence on $\hat{\phi}$ exist), and $\Im(\hat{S}) = \theta_i$ is a conserved quantity along i th “Morse flow”, also called steepest descent path. We will refer to $\hat{\mathbf{S}}$ as holomorphic action functional [8] which will play a significant role in constructing the flow equations next [24]. Finally, anything boldfaced is a *functional* below.

The paths are governed by the following set of equations:

$$\frac{\partial \hat{\phi}}{\partial \tau} = - \left(\frac{\delta \hat{\mathbf{S}}}{\delta \hat{\phi}} \right), \quad \frac{\partial \bar{\hat{\phi}}}{\partial \tau} = - \left(\frac{\delta \hat{\mathbf{S}}}{\delta \bar{\hat{\phi}}} \right) \quad (4)$$

It is important to note that the parameter τ is a *flow parameter* that may be independent of time direction. In general, we take $\tau \in \mathbb{R}$ and therefore the infinite dimensional complexified field space is defined to be $\Gamma_{\mathbb{C}} = \{\hat{\phi}(\mathbf{z}, t; \tau) | (\mathbf{z}, t; \tau) \in \mathbb{C}^{d-1} \times \mathbb{R}_t \times \mathbb{R}_\tau \equiv \mathbb{C}^d\}$. Note that in writing flow equations, we have specified the Riemannian metric of Γ to be flat. Using flow equations, we define

$$\frac{\partial \phi_R}{\partial \tau} \equiv \frac{1}{2} \left(\frac{\partial \hat{\phi}}{\partial \tau} + \frac{\partial \bar{\hat{\phi}}}{\partial \tau} \right) = -\frac{1}{2} \left(\frac{\delta \mathbf{h}}{\delta \phi_R} + \frac{\delta \theta_i}{\delta \phi_I} \right), \quad (5)$$

$$\frac{\partial \phi_I}{\partial \tau} \equiv \frac{1}{2i} \left(\frac{\partial \hat{\phi}}{\partial \tau} - \frac{\partial \bar{\hat{\phi}}}{\partial \tau} \right) = -\frac{1}{2} \left(\frac{\delta \mathbf{h}}{\delta \phi_I} - \frac{\delta \theta_i}{\delta \phi_R} \right). \quad (6)$$

Let us now focus on a simple harmonic oscillator in $1d$. In this case, we identify $\phi = \frac{1}{\sqrt{2}}(p + ix)$ where our particle's coordinate x and momentum p are implicitly t and τ -dependent and for simplicity we have set $m, \omega = 1$. Hence, the last forms of flow equations turn out to give

$$\frac{\partial p}{\partial \tau} = -\left(\frac{\partial \mathbf{h}}{\partial p} + \frac{\partial \theta_i}{\partial x}\right), \quad \frac{\partial x}{\partial \tau} = -\left(\frac{\partial \mathbf{h}}{\partial x} - \frac{\partial \theta_i}{\partial p}\right) \quad (7)$$

If we ask for implicit τ -dependence of x and p as well, we see that along with a Morse flow determined by the first terms, these equations coincide with Hamilton's equations along the i th flow line where up to a constant $H_i(\tau) = i\theta_i$ is the Hamiltonian flow of the system, which is conserved under flow with respect to τ : $dH_i/d\tau = 0$. There is a general way to see this using (4), which is,

$$dH_i/d\tau \propto \frac{d[\hat{S} - \tilde{S}]}{d\tau} = \int_{\mathbb{R}} dt \left(\frac{\delta \tilde{\mathbf{S}}}{\delta \hat{\phi}} \frac{\partial \hat{\phi}}{\partial \tau} - \frac{\delta \hat{\mathbf{S}}}{\delta \hat{\phi}} \frac{\partial \hat{\phi}}{\partial \tau} \right) = 0.$$

Let us see what $H_i(\tau)$ really looks like. If we set the canonical conjugate momentum to be

$$\Pi_\phi = \frac{-1}{\sqrt{2}}(ip + x),$$

(which satisfies classical Poisson bracket $\{\phi, \Pi_\phi\} = 1$), then the Hamiltonian flow in a harmonic potential correctly reads

$$\begin{aligned} H_i(\tau) &= \int_{\mathbb{R}} dt (\Pi_\phi^2 + \phi^2) = i \int_{\mathbb{R}} dt px \\ &:= i \int_{\mathbb{R}} dt \mathcal{H}_i(\tau, t) \equiv i\theta_i, \end{aligned} \quad (8)$$

where we have defined the real-valued “flowing” Hamiltonian of the system $\mathcal{H}_i(\tau, t)$ along i th flow. At every fixed τ this gives the real classical Hamiltonian of the system which only evolves in t .

Upon quantization, we have $[\phi, \Pi_\phi] = [x, p] = i$ and the Hamiltonian flow for a harmonic potential now reads

$$H_i(\tau) = \int_{\mathbb{R}} dt (\Pi_\phi^2 + \phi^2) = i \int_{\mathbb{R}} dt \frac{1}{2}(xp + px). \quad (9)$$

Therefore, $H_i(\tau) = i\theta_i = i \int_{\mathbb{R}} dt \frac{1}{2}(px + xp)$, where now x and p do not commute.

For the flowing Hamiltonian along the i th flow, i.e. $\mathcal{H}_i(\tau, t) := p(\tau, t)x(\tau, t)$, Hamilton's equation for x gives

$$\dot{x} = \frac{\partial \mathcal{H}_i(\tau, t)}{\partial p} = x \quad (10)$$

that has the unique solution $x(\tau, t) = (1/\sqrt{T})X(\tau)e^t$, for some smooth flow function X where $T = t_f - t_i$ is a normalization time scale. Similarly, $p(\tau, t) = (1/\sqrt{T})P(\tau)e^{-t}$, which proves the flowing Hamiltonian \mathcal{H}_i has no explicit t -dependence, as expected.

Plugging x, p back in (7) gives

$$H_i(\tau) = iP(\tau)X(\tau) \quad (11)$$

To find P and X we should solve flow equations (5) and (6). But before doing that, let us go back to (8) and plug it directly in (7) to get

$$H_i(\tau) = i \int_{x_i}^{x_f} p dx, \quad (12)$$

which, modulo the factor i , is what is sometimes called the “abbreviated action”. We recall that this Hamiltonian flow was considered to be the imaginary part of the original complexified action \hat{S} along the i th flow so no surprise why it is the integral of a tautological 1-form $\Theta = ipdx$ on the cotangent bundle T^*M of a 2-manifold M , the so-called phase space of the original $1d$ problem.

Introducing Hamilton's equations in the definition of Legendre transformation in Euclidean signature, we get

$$L = 2i\dot{\phi}\Pi_\phi - \mathcal{H}_i = -(p^2 - x^2) + ipx \quad (13)$$

and thus our holomorphized action in configuration space reads

$$\begin{aligned} \hat{S} &= \int_{\mathbb{R}} dt (-(p^2 - x^2) + ipx), \\ \mathbf{h} &= -(p^2 - x^2), \quad \theta_i = px. \end{aligned} \quad (14)$$

Therefore, flow equations (5)-(6) yield

$$\frac{dX(\tau)}{d\tau} = -X(\tau), \quad \frac{dP(\tau)}{d\tau} = P(\tau), \quad (15)$$

and solving these we get the *conserved* Hamiltonian flow as

$$H_i = iX_0^i P_0^i, \quad (16)$$

where (X_0^i, P_0^i) shows the initial values of the motion along the i th flow.

Now it is just left to determine what X_0 and P_0 are. Recall that if action \hat{S} is stationary at some point, we have from flow equations, $dX/d\tau = dP/d\tau = 0$ that gives rise to trivial solutions $X = X_0^i, P = P_0^i$ along i th flow line that passes through the critical point, say, $\rho_i = (x_i, p_i)$. But, indeed, the holomorphized harmonic oscillator problem still has the same critical point, $\rho = 0$ as before. Hence, in this specific case, we are to drop subscript i , and then from $\Im[\hat{S}(\rho)] = 0$, we deduce that our Hamiltonian flow H vanishes. So any particle that starts at the critical point, will remain there for all times. So we put $P_0 = 0$ but X still flows in τ , so does $x(t, \tau) = X_0 e^{t-\tau}$, which is not acceptable unless $t = \tau$. Then the path integral measure changes to dX_0 and everything will be consistent. The integration cycle is the whole real line which happens to be a *downward* flow line over which $h \rightarrow \infty$ as $\tau \rightarrow \infty$.

Therefore, in the $1d$ path space of harmonic oscillator Γ , the holomorphized path integral becomes

$$Z = \int_{\Gamma_{\mathbb{R}}} \mathcal{D}\phi e^{-S[\phi]} \rightarrow \int_{\Gamma_{\mathbb{C}}} \mathcal{D}\hat{\phi} e^{-\hat{S}[\hat{\phi}]} \quad (17)$$

$$(\text{along downward flow lines}) \rightarrow \int_{\mathcal{J}=\mathbb{R}} dX_0 e^{-\hat{S}[0, X_0]}.$$

This is the most basic example of *localization to constant paths* which happened in this case as $(x, p) \rightarrow (X_0, 0)$. Technically speaking, such flow line is called a ‘‘Lefschetz thimble’’ which we denote by \mathcal{J} and here the Lefschetz thimble simply coincides with \mathbb{R} . Lefschetz thimbles connect two good regions G_i to each other. Here, ‘good’ really means that the integral converges in these regions. There is a homological interpretation of this idea that will come in a bit.

However, the opposite scenario would have led to an *upward* flow - also known as \mathcal{K} -cycle over which $h \rightarrow -\infty$. This is not good because a \mathcal{K} -cycle always flows in orthogonal directions to \mathcal{J} -cycles and therefore the integral always diverge on them. This orthogonality means that in general, the intersection of a \mathcal{K}_i -cycle and a \mathcal{J}_j -cycle happens to give either one or zero. This in compact notation is written as $\langle \mathcal{K}_i, \mathcal{J}_j \rangle = \delta_{ij}$. It may be worth mentioning that a downward flow that starts at one critical point cannot end at another critical point and will always flow to $h \rightarrow \infty$ unless we are on a ‘‘Stokes ray’’ where flow lines overlap. On this line, the \mathcal{K}_i and \mathcal{J}_i -cycles do not intersect each other.

Given a (non-compact) manifold X of arbitrary dimension, if for some very large flow time $\tau < \tau^*$, $h > L$ with L being really large, then X_{τ^*} denotes the union of good regions G_i . So we can form \mathbb{Z} -valued relative real k th-homology group $H_k(X, X_{\tau^*}; \mathbb{Z})$ and use a Morse function h on X to determine an upper bound for the rank of each group. It can be shown that if the differences between the Morse indices of distinct critical points of h are different from ± 1 , then the ranks of the homology groups are equal to the upper bounds m . In our trivial example, the Morse index of $\rho = 0$ is one so $H_1(X, X_{\tau^*}; \mathbb{Z})$ is of rank 1 and all other relative homology groups vanish. In the context of supersymmetric theories in physics, the invariant measured by this relative homology is usually a Witten-type index, which by a clue from the bosonic harmonic theory discussed above, roots back to localization ideas.

In general, this holomorphization procedure for any path integral over a field space $\Gamma_{\mathbb{R}}$ yields an equivalent formulation with a path integral over an integration cycle in the complexified field space $\Gamma_{\mathbb{C}}$, such that we can write the path integral as

$$Z = \int_{\Gamma_{\mathbb{R}}} \mathcal{D}\phi e^{-S[\phi]} \equiv \sum_{i \in \Sigma} n_i \int_{\mathcal{J}_i} \mathcal{D}\hat{\phi} e^{-\hat{S}[\hat{\phi}]} \quad (18)$$

where $n_i \in \{\pm 1, 0\}$ and Σ is the set of all critical points of \hat{S} . This is what one means by Picard-Lefschetz theory.

As a last remark, we have to point out that in case the critical points are degenerate such that their Morse index is ill-defined, or equivalently if the (effective) action is ‘‘resonant’’, we do not give up on the formula (18) but take the degeneracy factor into consideration by asking n_i to take values in \mathbb{Z} , instead. This factor is technically undefined and needs regularization since $1/\mathcal{H}(p) = \pm \infty$

where $\mathcal{H}(p)$ is the Hessian at the degenerate saddle point p . However, the degeneracy can be removed topologically by path homotopy equivalence of flow lines. In short, any two flow lines attached to degenerate (and non-isolated as in gauge theories) saddle points that *belong* to the same homotopy class are deemed equivalent and should be counted only once. Homologically, the \mathcal{J} -cycles in $H_1(X, X_{\tau^*}; \mathbb{Z})$ must describe a basis for any generic cycle γ in X with its boundaries lying in X_{τ^*} .

Even though in a theory with degenerate saddle points, a Morse function is ill-defined (unless as shown in subsection [5] regularized in a way that degeneracy is lifted), still we can distinguish between saddle points and corresponding Lefschetz thimbles by path homotopy. This enables us to define the irreducible \mathbf{J} -set whose elements \mathcal{J}_i are (1) generators of $H_1(\mathfrak{F}, \mathfrak{F}_{\tau^*}; \mathbb{Z})$ and (2) form a homotopy class $[\mathcal{J}_i]_{\pi}$ where $[\cdot]_{\pi}$ means ‘modulo path homotopy’. This removes the degeneracy of saddle points in theory and renders a clearer interpretation of complex Morse function analysis of the homological quantum mechanics in the presence of a gauge group provided that the homology coefficients be determined by Witten index check or other alternative checks discussed in [5.3]. We will concentrate on the imaginary phase of this complex Morse function at the saddle rims defining the elements of \mathbf{J} -set in our treatment of Picard-Lefschetz theory.

3. HIDDEN TOPOLOGICAL ANGLES, SPECTRAL CURVES AND NON-BPS OBJECTS

Eq. (12) in general defines hidden topological angles in the phase space of all paths described by flow equations (4). By following a periodic trajectory in phase space of a harmonic oscillator, $x_i = x_f$, this equation in units of \hbar gives

$$H_i = i \oint_{\gamma} p dx = 2\pi i k_i \Rightarrow \theta_i = 2\pi k_i, \quad (19)$$

for integer k_i and γ being a circle of radius $\sqrt{2E}$ at energy level E .

Yet another method to get the phases of saddle points would be through studying algebraic geometry and (co)homology theory of the potential curves. In (non)supersymmetric theories with double-well superpotential $W = z^3/3 - z$ [12], the only non-perturbative contribution to ground state energy comes from a non-BPS exact complex instanton-antiinstanton solution which starts at the true vacuum of the system at z^m and turns back from either the complex turning point z^T or its complex conjugate \bar{z}^T , completing the periodic motion with infinite periodicity. This system is part of a bigger family of integrable systems which are described by the singular spectral curves $\mathfrak{S}_i(\mathbb{C})$ defined (in Euclidean signature) by

$$\begin{aligned} y_i^2 &= (W'(z))^2 + k_i g W''(z) + 2E_i^m \\ &= (z^2 - 1)^2 + 2k_i g z + 2E_i^m \end{aligned} \quad (20)$$

where $g \in \mathbb{R}$ is a coupling constant, k_i is the number of fermionic degrees of freedom, and $-E_i^m$ is energy at the global minimum of the potentials $V_i(z) = \frac{1}{2}(W'(z))^2 + \frac{k_i}{2}gW''(z)$ with $W'(z)$ being of at least degree 2. In general, k_i may be an integer but only for $k_i = 1$ (or $k_i = -1$ in certain cases) the theory is supersymmetric. For double-well superpotential, the maps (20) look like $y^2 \sim (12(z_i^m)^2 - 4)(z - z_i^m)^2$ near $z_i^m \in \mathbb{R}$, so z_i^m have multiplicity 2 and since they are global minima of potential graphs, hence simple ramification points of index $\nu_{z_i^m} = 2$. The first relative (co)homology group of the spectral curves (20) with respect to the ramification divisor $\mathcal{D} = 2z_i^m$ is $\mathbb{Z}/2\mathbb{Z}$ -graded which is shown by computing the integral of the 1-form $\sqrt{y^2}dz$ over the contour \mathcal{C} that encircles a complex conjugate pair of fixed points, e.g. turning points, $[z^T, \bar{z}^T]$ of $\mathfrak{S}_i(\mathbb{C})$ at energy level E_i^m :

$$H_i = -\frac{1}{g} \int_{-\Im(z^T)}^{\Im(z^T)} dy \sqrt{2E_i^m + W'(z)^2 + k_i g W''(z)}, \quad (21)$$

where $z \equiv \Re(z^T) + iy$, which is literally nothing but the phase of WKB approximation at zero order in g -expansion. This integral for a genus zero singular algebraic curve is calculated to be $H_i = i\pi k_i$, giving what we had called early on a hidden topological angle (HTA): $\theta_i = \pi k_i$ [12]. A motivation for this name comes from the disability to see explicitly any input in flow equations (4) that includes a topological term to produce *a priori* a phase for the saddle points.

We now seek a set of appropriate transformations to reduce $\mathfrak{S}_i(\mathbb{C})$ with $W = z^3/3 - z$ to nodal cubic curves and comment on the connection between singular points and HTA.

Starting with the generic algebraic curve

$$y_i^2 + a_1^i zy + a_3^i y - (z^3 + a_2^i z^2 + a_4^i z + a_6^i) = 0, \quad (22)$$

where

$$a_1^i = \frac{2k_i g}{\sqrt{1+2E_i^m}}, a_2^i = -2 - \frac{k_i^2 g^2}{1+2E_i^m}, \\ a_3^i = 0, a_4^i = -4(1+2E_i^m), a_6^i = 4k_i^2 g^2 + 8(1+2E_i^m),$$

we find that the set of transformations

$$z = \frac{1}{u^2} (2\sqrt{1+2E_i^m}(v + \sqrt{1+2E_i^m}) + 2k_i gu) \\ y = \frac{1}{u^3} (4(1+2E_i^m)(v + \sqrt{1+2E_i^m}) \\ + 2\sqrt{1+2E_i^m}(2k_i gu + cu^2) - \frac{d^2 u^2}{2\sqrt{1+2E_i^m}}),$$

turn (22) into the quartic equation $v^2 = 2E_i^m + 2k_i gu + (u^2 - 1)^2$ describing $\mathfrak{S}_i(\mathbb{C})$ in (u, v) coordinates. Hence, the algebraic curves

$$y^2 - z^3 + (2 + \frac{k_i^2 g^2}{1+2E_i^m})z^2 + z(4 + 8E_i^m + \frac{2k_i g}{\sqrt{1+2E_i^m}}) \\ - 8 - 16E_i^m - 4g^2 k_i^2 = 0, \quad (23)$$

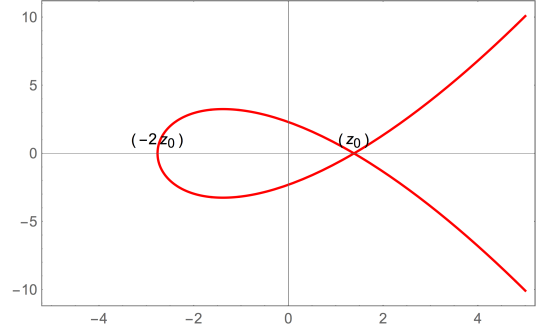


Figure 2. Graph of the reduced spectral curves $\mathfrak{S}_i(\mathbb{C})$ over the $\text{char} = 0$ field of complex numbers. The closed orbit here shows an exact periodic non-BPS solution of equations of motion in the tilted double-well potential known as *complex bion* [IT]. There is an ordinary double point, i.e., node, at z_0 originated from the singularity of the curves (20). Unlike smooth elliptic Weierstrass curves, $\mathfrak{S}_i(\mathbb{C})$ have genus 0 in view of Riemann-Hurwitz formula.

are the reduced spectral curves associated with degree 4 orbits at energy E_i^m in the inverted double-well potential. Introducing

$$b_1^i = (a_1^i)^2 + 4a_2^i, \\ b_4^i = a_1^i a_3^i + 2a_4^i, b_6^i = a_3^i + 4a_6^i,$$

and a change of variables $y \rightarrow 2y$ and $z \rightarrow z - \frac{b_1^i}{12}$, equation (23) turns into

$$y^2 - z^3 + (\frac{16}{3} + 8E_i^m)z - 4k_i^2 g^2 - \frac{32E_i^m}{3} - \frac{128}{27} = 0. \quad (24)$$

Numerical analysis verifies that the discriminant of this algebraic cubic curve is zero because of an ordinary double point at z_i^m :

$$\Delta = 64(\frac{16}{3} + 8E_i^m)^3 - 16(108k_i^2 g^2 + 288E_i^m + 128) \\ = 0, \quad (25)$$

from which one can also solve for the energy of complex bion [IT] correlated events. Same analysis indicates that there is a simple turning point at $z = -2z_0$ which is a branch point of the spectral curve (24), and a double turning point at $z = z_0$, being in turn the ordinary double point of $\mathfrak{S}_i(\mathbb{C})$, see fig. [2]. Therefore, the final form of the algebraic curve encoding complex bion can be written as the nodal cubic curve

$$E_0 : y^2 = (z - z_0)^2(z + 2z_0). \quad (26)$$

The genus of this curve is 0 and over \mathbb{C} it indeed is the Riemann sphere or complex projective space $\mathbb{P}_1(\mathbb{C})$ with the two poles $z = 0, \infty$ identified. Technically, a torus with a complex structure constant \mathbb{k} , $T^2 = \mathbb{C}/(\mathbb{Z} + \mathbb{k}\mathbb{Z})$ becomes homeomorphic to a nodal cubic curve at the limit $\mathbb{k} \rightarrow i\infty$ as one cycle goes away. There is a holomorphic map $\mathbb{P}_1(\mathbb{C}) \rightarrow \mathbb{P}_2(\mathbb{C})$ defined by $h : w \mapsto [z(w), y(w), 1]$

where

$$z(w) = z_0 + \frac{12wz_0}{(w-1)^2}, \text{ and } y(w) = \frac{4w(w+1)(3z_0)^{\frac{3}{2}}}{(w-1)^3}, \quad (27)$$

that maps $\mathbb{P}_1(\mathbb{C})$ onto nodal/singular curve E_0 which is injective away from the points $0, \infty \in \mathbb{P}_1(\mathbb{C})$ at which the double point singularity of E_0 , namely, $[z_0, 0, 0]$ is reached via h .

The orbit seen in fig. [2] is a homology cycle (a line generated as a result of flow by steepest descent) attached to the north pole (critical point) of S^2 , starting there at time $\tau = -\infty$, and finally reaching the south pole at time $\tau = \tau^* = \infty$. In spherical coordinates, this flow line is actually a semi great circle $\phi = \phi_0$ with the conserved quantity being

$$\text{circumference} = \pi k_i \quad (28)$$

where k_i is radius of the Riemann sphere for \mathfrak{E}_i . This is exactly equal to the HTA calculated using the integral formula given in (21). Upon taking k_i to be an integer, we get a Bohr-Sommerfeld type quantization for the HTA and no resurgence will take place in the sense of [13]. Thus, the orbit in fig. [2] clearly represents the circle of $E_0 \cong S^2 \vee S^1$. It is hence expected that any quantum theory with E_0 as its quantized moduli space of classical vacua accepts a possible non-BPS exact solution contributing non-perturbatively to the vacuum energy and breaking supersymmetry.

More insight in this regard is drawn from the thimble analysis of near-supersymmetric $\mathbb{P}_{k-1}(\mathbb{C})$ quantum mechanics in the quasi moduli space of kinks and anti-kinks [25]. There it is demonstrated using explicit calculations for $k = 2$ case that the theory does entail complex bion solutions but they do not have a mod π HTA. We elucidate this result in the context of current section by mentioning that the length of every great circle of $\mathbb{P}_1(\mathbb{C}) \cong S^2$ corresponding to a non-perturbative solution in the supersymmetric limit (i.e. $\epsilon = 1$ in [25]) is naturally equal to 2π in units of ϵ with an integer winding number for higher order (real) bions. Also, it is obvious that the spectral curve $y = \sqrt{V(|z|) + 2E^m}$ does not develop a cusp singularity at the energy level $-E^m$ associated with global minimum of the potential $V(|z|)$ because y only depends on $|z| \geq 0$ ($|z|$ being the modulus of the inhomogeneous coordinate z) contrary to systems $\mathfrak{E}_i(\mathbb{C})$ described by (20).

If we insist to know the Picard-Lefschetz theory of a complex bion, it is apparent that the singularity in fig. [2] is a setback. But it is a cusp catastrophe that could be well examined using catastrophe theory which replaces Picard-Lefschetz theory as the complex bion occurs as a degenerate saddle point in (near)supersymmetric quantum theories (cf. fig. 26 of [8]). It would be nice to understand it, but we leave that to a future work.

Higher genus algebraic curves that develop complex bions $[\mathcal{I}\mathcal{I}]^n$ ($n \geq 1$) are also obtained by generalizing $\mathfrak{E}_i(\mathbb{C})$ to higher degree ($d > 4$) spectral curves with singular points. Note that on smooth curves, the genus formula

is the dimension of space of holomorphic 1-forms i.e., $\frac{P_j(z)dz}{y}$ for a polynomial(s) $P_j(z)$ of degree $\leq d-3$ on $\mathfrak{E}_i(\mathbb{C})$. However, due to singular points $\mathcal{S} = \{z_1, \dots, z_\ell\}$ in $\mathfrak{E}_i(\mathbb{C})$, genus formula is replaced by the dimension of sheaf cohomology of holomorphic 1-forms, defined by

$$\mathbf{g} := \dim_{\mathbb{C}} H^0(\mathfrak{E}_i, \Omega_{\mathfrak{E}_i}) = \dim_{\mathbb{C}} H^1(\mathfrak{E}_i, \mathcal{O}_{\mathfrak{E}_i}) \quad (29)$$

following Serre duality $H^1(\mathfrak{E}_i, \mathcal{O}_{\mathfrak{E}_i}) \cong H^0(\mathfrak{E}_i, \Omega_{\mathfrak{E}_i})$, where $\mathcal{O}_{\mathfrak{E}_i}$ is the sheaf of holomorphic functions and $\Omega_{\mathfrak{E}_i}$ being the sheaf of holomorphic 1-forms for which $P_j(z_{1 \leq i \leq \ell}) = 0$. Note that here the sheaf cohomology is basically derived based on an inclusion mapping of polynomials $P_j(z)$ of degree $\leq d-3$ with singular divisor $\mathcal{D} = z_1 + \dots + z_\ell$ into generic polynomials of degree $\leq d-3$ where in the former, \mathcal{S} imposes independent linear conditions on the coefficients of the polynomials of degree $\leq d-3$. These conditions are necessary and *usually* sufficient to reduce the “naive” genus of \mathfrak{E}_i to that of desingularization of it, $\bar{\mathfrak{E}}_i$.

Local sections of $\Omega_{\mathfrak{E}_i}$ give corrections to the cohomology classes of the (classical) “WKB” 1-form $\omega = g^{-1} \sqrt{y^2} dz$ in theories with $\mathbf{g} > 0$. By cohomology class we mean a differential of the form (modulo an overall sign factor)

$$\tilde{\omega} = \omega + \sum_{j=1}^{\mathbf{g}} n_j \frac{P_j(z)dz}{y}, \quad (30)$$

on $\mathfrak{E}_i(\mathbb{C})$ that assigns a number to each cycle of real codimension 1 that measures the non-triviality of those cycles in failing to vanish. This number is the HTA upon specifying the cycle. Let us mention that the coefficients n_j are not arbitrary and depend on a number of factors such as barrier width of the potential curve that will be investigated with explicit examples in a future work.

Such cycles in WKB cohomology, that are physically relevant, are basically closed curves that go around the same non-trivial complex conjugate fixed points of WKB 1-form, which in the process are exposed to cycles in the sheaf cohomology of holomorphic 1-forms -thus by Serre duality to sheaf cohomology of holomorphic functions- which roughly speaking, can be explored by looking directly at the contour plots of holomorphic 1-forms and those of ω at a given energy, say E_i^m . If there is an overlap between these cycles, then there should be a correction to the WKB result at zeroth order in g -expansion as prescribed in eq. (30), which is totally overlooked in the classical treatment of singular spectral curves. In fact, this method does not capture the full topology of singular algebraic curves but only explains it partially via the information hidden in the fixed points (turning points) of the WKB 1-form. Hence, the sheaf of holomorphic 1-forms $\Omega_{\mathfrak{E}_i}$ decides on how topology change in $\mathfrak{E}_i(\mathbb{C})$ would correct the phase angle of WKB solution when turning points are complex and quantum corrections come from the higher order terms in g -expansion. This is however not the case for an smooth affine curve, where topological recursion has been proven to indeed reconstruct the WKB expansion of the quantum spectral curve [26].

4. INDEX FORMULA FROM PICARD-LEFSCHETZ THEORY: $\mathcal{N} = 4$ QUANTUM MECHANICS

Reminder: In the rest of this paper, we will adopt the notation $\kappa = 2i\pi$ for simplicity. Our aim in this section is to write down a formula like (18) that correctly computes the refined (categorified) Witten index

$$\mathcal{I}(y, \zeta) := \text{tr}(-1)^F e^{-\beta H} y^J \quad (31)$$

for the supersymmetric quantum mechanics with four real supercharges, and a $U(1)$ R -symmetry J that appears as a left-moving $U(1)$ R -symmetry in the original theory that will be taken to be $\mathcal{N} = (2, 2)$. Excited states will not be playing a role in the quasi-topological index we are seeking in what follows so there will not be any dependency on β anywhere. The index depends on both the R -charges of chiral fields and the Fayet-Iliopoulos (FI) parameters ζ . The latter enter the index formula in view of a Q -exact deformation term in the $\mathcal{N} = (2, 2)$ Lagrangian. Hence, Witten index is categorized in terms of the R -symmetry gauge holonomy y defined by

$$y = \exp \frac{1}{2} \kappa \left(\oint_{S^1} A_R \right) \equiv e^{\frac{1}{2} \kappa z}, \quad (32)$$

where $A_R := \frac{z}{2\pi}$ is the $U(1)_R$ gauge field.

This formalism will totally avoid the JK residue operation used in [20] since we are considering relative homology cycles \mathcal{J}_i as constituent components of the closed integration cycle γ and focus on the thimble integrals within the fundamental domain of the path space X -interchangeably called u -space from now on- parametrized by

$$u_i = \oint_{S^1} A_i^{\text{flavor}} + ix, \quad x \in \mathbb{R} \quad (33)$$

where A_i^{flavor} is the i th $U(1)$ flavor gauge field of the dimensionally reduced $2d$ theory that can be taken as an $\mathcal{N} = (2, 2)$ theory on a torus T^2 . x is a real constant scalar that (together with A_i^{flavor}) parametrizes the moduli space of the supersymmetric configurations for the $\mathcal{N} = (2, 2)$ gauge theory. The anti-holomorphic variables are washed away from the 1-loop determinant after taking the limit $D \rightarrow 0$ where D is the auxiliary field in the chiral multiplet [16].

The quantum mechanics in consideration has the total gauge group $G = U(1)^\alpha$ unless otherwise stated ¹. Since

the chiral matter is chosen to be in the bifundamental representation of G , allowing an overall $U(1)$ to be decoupled in the process, arrows will indicate the bifundamentals in a quiver graph and nodes will represent a $U(1)$ quiver gauge group.

We now generalize the discussion of non-compact path space of $U(1)$ gauge theory and its constituent Lefschetz thimbles in section [1] to higher-rank gauge groups. The general premise then is that only the saddle points of the effective 1-loop determinant at imaginary-infinity will contribute to the χ_y genus and the corresponding refined Witten index of the $1d$ gauged quiver system. From Picard-Lefschetz theory, χ_y genus can be written down as²

$$\begin{aligned} \mathcal{I}(z, \zeta) &= \frac{1}{\kappa^\alpha} \sum_{j=1}^{2^\alpha} n_j(\zeta) \int_{\mathcal{J}_j} \mathcal{Z}^{1L} d\vec{u} \\ &= \frac{1}{\kappa^\alpha} \sum_{j=1}^{2^\alpha} n_j(\zeta) \lim_{\vec{u} \rightarrow \vec{u}_j^*} \mathcal{Z}^{1L}(y, \vec{u}) \\ &\equiv \frac{1}{\kappa^\alpha} \sum_{j=1}^{2^\alpha} n_j(\zeta) Z_j(y). \end{aligned} \quad (35)$$

where $d\vec{u} = \prod_a du_a$, α is the total number of $U(1)$ factors in G of the quiver system, i.e., rank of G , 2^α corresponds to the total number of saddles distinguished by the integration of \mathcal{Z}^{1L} over j th Lefschetz thimble, \mathcal{J}_j , attached to the j th saddle point at \vec{u}_j^* . The number of Lefschetz thimbles is determined by the number of different limits at imaginary infinity directions. For example, for $G = U(1)^4$, one finds $\vec{u}_j^* = i\vec{v}_j\infty$ where \vec{v}_j is a 4-vector with components $+1$ and/or -1 , and correspondingly there are $2^4 = 16$ \mathcal{J} -cycles that construct the integration cycle. The limits in (35) give the thimble integrals exactly that themselves measure a relative homology as discussed in section [2].

On supersymmetric grounds, as in this paper, if the moduli space \mathcal{M} is compact, one can write a refined index formula of the form

$$\mathcal{I}(y, \zeta) = \sum_{p,q=0}^D h_{(p,q)}^{\mathcal{M}}(\zeta) (-1)^{p-q} y^{2p-D}. \quad (36)$$

Here, we are talking about a compact moduli space \mathcal{M} of complex dimension D endowed with a Kähler structure that entails all we need about the ground state spectrum of the underlying theory. The numbers $h_{(p,q)}^{\mathcal{M}}$ are the

¹ A generic group in the literature of gauged quiver quantum mechanics happens to mostly be of the form $G = \prod_i U(n_i)$ for $n_i \in \mathbb{Z}^+$ where n_i are the coefficients for the contribution of each elementary BPS state charge c_i^Q present in the $\mathcal{N} = 4$ quiver quantum mechanics to the BPS charge of a one-particle $4d$ $\mathcal{N} = 2$ system [27]:

$$c^{4d} = \sum_i m_i c_i^Q. \quad (34)$$

The numbers m_i determining the basis for the space of BPS particles *should* coincide with the magnitude of relative homology coefficients that is built out of flow lines satisfying equations (4) with the effective 1-loop action of the $\mathcal{N} = 4$ quiver system. In this work, we will proceed to verify that the flow diagram in the u -space accurately captures the BPS configuration contributing to the index with m_i as given in (34).

² In case of having $\#$ decoupling $U(1)$'s, one has to set $\alpha \rightarrow \alpha - \#$.

Hodge numbers of the bigraded algebra corresponding the Dolbeault cohomology groups. A closer look to our index formula (35) suggests that it actually is similar in nature to (36). In Picard-Lefschetz theory, the homology cycles \mathcal{J} are labeled “relative” $H_1(X, X_{\tau^*}; \mathbb{Z})$ for large flow time in non-compact manifolds X containing singularities³, whereas the cohomology of \mathcal{M} in (36) is global. So a direct comparison might not be possible between $h_{(p,q)}^{\mathcal{M}}$ and intersection numbers. However, we will see that a slight modification of this last formula would make it feasible to find a relation between the two in the next section.

5. LINEAR ABELIAN QUIVER QUANTUM MECHANICS (DYON CHAINS)

Physically, quiver quantum mechanics appears as the low energy theory of a set of wrapped D -branes which encodes information on the dynamics of single and multicentered BPS black hole geometries in $4d \mathcal{N} = 2$ supergravity. It also arises in $2d \mathcal{N} = (2, 2)$ gauge theories. The gauged quantum mechanics governs the low energy sector of the full quantum field theory, which is then useful for index calculations and understanding the dynamics of BPS bound states. A famous example that has been studied heavily in [28, 29] is dyon chains that will be our focal point in the rest of this paper.

2-node quiver: $\mathcal{N} = 4 \mathbb{P}_{k-1}(\mathbb{C})$ model

Let us study the 2-node linear supersymmetric quiver quantum mechanics also known as $\mathbb{P}_{k-1}(\mathbb{C})$ derived from $4d \mathcal{N} = 2 SU(2)$ Yang-Mills theory. It is a massive rank-one Abelian theory that does not flow to a fixed point at IR and hence is not superconformal. Therefore, the R -symmetry fugacity is discrete (as $z \in \mathbb{Z}/k$). There are k bifundamental chiral fields between the nodes, represented by k -overlapping arrows in fig. [5].

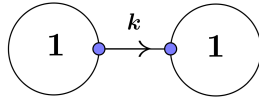


Figure 3. A $\mathbb{P}_{k-1}(\mathbb{C})$ model. These are 2-node linear quivers with rank $\alpha = 1$ Abelian gauge groups at each node. k , the number of arrows, also shows the number of bifundamentals.

³ The quality of being relative for a good cycle means that for any cycle $\mathcal{J} \in X$ for some punctured non-compact manifold X , we need to find its ends at large flow time limit $\tau \rightarrow \tau^*$ in good regions $G_i, G_j \in X_{\tau^*}$ for $i \neq j$ where X_{τ^*} is defined to contain all good regions relative to the whole of X accessible at large flow time.

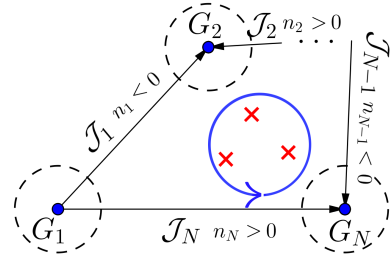


Figure 4. To fix the orientation for each thimble we start from one good region and continue counter-clockwise toward the other good regions until we come back to the starting good region. In other words, $\gamma = n_1 \langle G_1 | G_2 \rangle + n_2 \langle G_2 | G_3 \rangle + \dots + n_{N-1} \langle G_{N-1} | G_N \rangle + n_N \langle G_N | G_1 \rangle$ where the sign of n_i makes the orientation of $\langle G_i | G_{i+1} \rangle \equiv \mathcal{J}_i$ -cycle align that of gamma. This is basically similar to Kirchhoff’s loop rule for a given electric network.

5.1. General Strategy

Let good regions G_i be open discs of radius $\epsilon \ll 1$ centered at the roots of the 1-loop determinant. We denote by $\langle G_i | G_j \rangle$ with $i < j$ any possible good cycle in the slightly extended domain $\mathfrak{E} := \{u \in X \mid -1 \leq \Re(u - z) < 1\}$ compared to the fundamental one, and aim to count the number of \mathcal{J} -cycles contributing to the integration cycle γ so that the χ_y genus captures the correct spectrum of BPS states.

In dealing with situations in which integration cycle γ is closed, on the contrary to open integration cycles, one needs to assemble a means of identifying orientations for Lefschetz thimbles because they do not have natural orientations. The result of this integral leads to an algebraic invariant -here a polynomial in the fugacity of $U(1)_R$ gauge field, y - that measures the relative homology $H_1(X, X_{\tau^*}; \mathbb{Z})$ on the space $X = \mathbb{C}^\times \cong \mathbb{C} \setminus \{0\} \cong \mathbb{C}/\mathbb{Z}$ where \cong means homeomorphic⁴. The point $u = 0$ coincides with a *fundamental* pole of \mathcal{Z}^{1L} and the \mathbb{Z} is taken to be the subgroup of translations by integer multiples of the fundamental pole that will constitute the set of poles of \mathcal{Z}^{1L} on the real u -direction. Computing this integral is usually a burdensome task for even a simple space like \mathbb{R}^2 , let alone more complicated manifolds of higher dimensions. But the key tool that allows us to do computations in the current work is the exactness of saddle point approximation thanks to localization. Above all else lies the fact that there are many finite saddles that do not contribute to the path integral, letting us only focus on those cycles that pass through the rims at infinite imaginary directions of u -space. Considering the fact that we know *a priori* that thimble integrals would yield expressions that have the same overall sign in terms of powers of y , we prescribe the rule pictured in fig. [4].

⁴ For a quantum mechanical quiver system of rank- α gauge group G , the u -space is simply $X = (\mathbb{C}^\times)^\alpha$ [20].

The starting point is to plot the solutions to flow equation (4) for the following effective action:

$$S_{\text{eff}}(u, y) := -\ln(\mathcal{Z}^{1\text{L}}). \quad (37)$$

To do this, we first solve the flow line equations (4) and count the possible number of good cycles in \mathfrak{E} . It turns out that this number is 6 as seen in fig. [5]. But two of them are not independent: $\langle G_1|G_3 \rangle$ through B_1 and B_2 , the green and black cycles, respectively. They can be decomposed as

$$\langle G_1|G_3 \rangle = \langle G_1|G_2 \rangle + \langle G_2|G_3 \rangle,$$

without wrapping any of the poles (red points). Naively, in total, there are 4 independent \mathcal{J} -cycles forming a γ , encircling two poles at $(0,0)$ and $(-1,0)$ in \mathfrak{E} . In other words, $[\langle G_1|G_3 \rangle] = [\langle G_1|G_2 \rangle \star \langle G_2|G_3 \rangle] = [\langle G_1|G_2 \rangle] \cdot [\langle G_2|G_3 \rangle]$ induced by the group operation \cdot of homotopy group $\pi_1(\mathfrak{E}; \mathbb{Z})$ and the corresponding path composition \star .

This is, however, not all there is to the story. $\langle G_1|G_2 \rangle$ and $\langle G_2|G_3 \rangle$ both share the same imaginary part of the saddle region at $\pm i\infty$, meaning that no matter what the real part of associated saddles are, the Lefschetz thimble integral will yield the same result over $\langle G_1|G_2 \rangle$ and $\langle G_2|G_3 \rangle$ ⁵. So in the complex u -plane, the saddle points $i\infty + r_{ij}$, attached to cycles $\langle G_i|G_j \rangle$ for $|i-j| < 2$, are only distinguishable for infinite r_{ij} , which for a bounded \mathfrak{E} is not feasible⁶. Similarly, saddle points $-i\infty + r_{ij}$ are not distinguishable for finite $-i\infty + r_{ij}$. Therefore, $\langle G_1|G_2 \rangle$ and $\langle G_2|G_3 \rangle$ through B_1 or B_2 regions, are indeed copies of the same cycles. This will reduce the total number of independent cycles to 2. We will call this the *irreducible J-set*, whose cardinality is 2^α . The cycles in \mathbf{J} are linearly independent and all the Lefschetz thimble integrals over \mathbf{J} form a vector space of dimension 2^α . Finally, we observe that this set consists of all the possible Lefschetz thimbles in the fundamental domain $\mathfrak{F} := \{u \in \mathbb{C}^\times | 0 \leq \Re(u-z) < 1\}$ where $\mathfrak{F} \hookrightarrow \mathfrak{E}$, i.e., \mathfrak{F} is only a retract of \mathfrak{E} . This is best understood by examining the saddle points of the 2-node quiver effective action (37) and its saddle points, which, too, are those of the 1-loop determinant. We show this in fig. [6].

Compared to JK-residue operation, a big advantage comes into play when we forget about the poles and the number of poles involved. Indeed, this comes up in

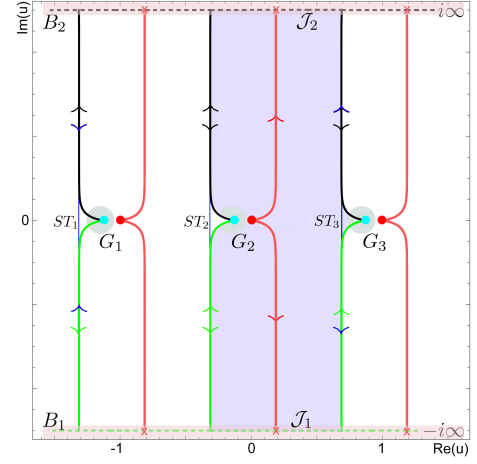


Figure 5. The plot of flow lines for $\mathbb{P}_{k-1}(\mathbb{C})$ model. Here, $k = 5$ and z is offset from $-2/5$ by 0.015 better observation of the flow lines. $B_1 \subset \mathfrak{B}_-$, $B_2 \subset \mathfrak{B}_+$ are bad regions in the imaginary-infinity locus subsets \mathfrak{B}_\pm where the relevant saddles are located. So every \mathcal{J} -cycle starting from either one of good regions, i.e., G_1, G_2, G_3 will flow along $\pm i\infty$ directions and end back in those regions. The lines labeled by ST_i represent Stokes rays on which \mathcal{J} -cycles overlap. The cross signs label the intersection points. The blue arrows belong to the cycles that flow from the domain \mathfrak{E} to a good region out or vice versa. Here, $n_1 = -n_2 = 1$. The integration cycle as seen in the form of elliptic genus is closed, which is in turn enforced by the orientation of the $\mathcal{J}_1, \mathcal{J}_2$ in γ , and thus by the intersection numbers.

our analysis naturally by just focusing on the hunt for an irreducible \mathbf{J} -set, which in the current case is $\mathbf{J} = \{\mathcal{J}_1, \mathcal{J}_2\}$, both being $\langle G_2|G_3 \rangle$ cycles (through B_1 and B_2 , respectively). This is shown in fig. [5]. For an Abelian 2-node quiver model with k bifundamentals, we have the 1-loop determinant form

$$\mathcal{Z}^{1\text{L}} du = -\pi \sin(\pi z)^{k-1} (\cot(\pi z) - \cot(\pi u))^k du \quad (38)$$

Since $u_1^* = -i\infty$ and $u_2^* = i\infty$, the thimble integrations over $\mathcal{J}_1, \mathcal{J}_2$ are computed by

$$\begin{aligned} Z_1(y) &= \lim_{u \rightarrow -i\infty} \frac{-\pi (\cot(\pi z) - \cot(\pi u))^k}{\sin(\pi z)^{1-k}} = \kappa \frac{y^{-k}}{y^{-1} - y}, \\ Z_2(y) &= \lim_{u \rightarrow i\infty} \frac{-\pi (\cot(\pi z) - \cot(\pi u))^k}{\sin(\pi z)^{1-k}} = \kappa \frac{y^k}{y^{-1} - y}. \end{aligned}$$

Therefore, the formula (35) produces the following index for the 2-node quiver quantum mechanics:

$$\mathcal{I}(y, \zeta) = n_1(\zeta) \frac{y^{-k}}{y^{-1} - y} + n_2(\zeta) \frac{y^k}{y^{-1} - y} \quad (39a)$$

$$\begin{aligned} &= \Theta(\zeta) \frac{y^{-k}}{y^{-1} - y} - \Theta(\zeta) \frac{y^k}{y^{-1} - y} \\ &= \begin{cases} \frac{y^{-k}}{y^{-1} - y} - \frac{y^k}{y^{-1} - y} \equiv y^{1-k} \sum_{i=0}^{k-1} y^{2i}, & \text{if } \zeta > 0 \\ 0, & \text{if } \zeta < 0 \end{cases} \quad (39b) \end{aligned}$$

⁵ A note is due to clear a possible misleading point here. The relative homology cycles can happen to actually yield the same invariant after thimble integration over them. Thus, two independent \mathcal{J} -cycles might in fact have an identical invariant, and yet be considered in the index formula. In general, the number of independent \mathcal{J} -cycles is 2^α where α is the rank of G or the number of nodes because $G = U(1)^\alpha$.

⁶ We note that the boundedness of \mathfrak{E} along real direction is necessary for the applicability of Picard-Lefschetz theory in this situation. If one lifted the boundedness requirement, the Lefschetz integration would be undefined for $\Re(u) \rightarrow \pm\infty$.

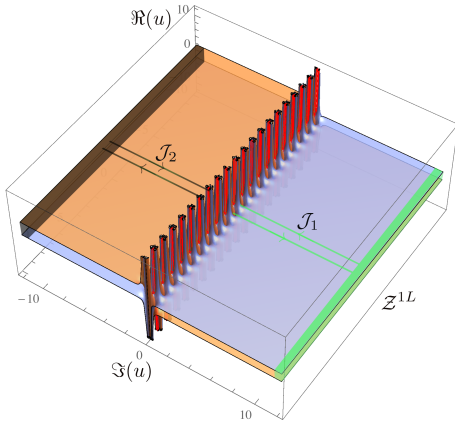


Figure 6. Real (violet) and imaginary (orange) parts of $\mathcal{Z}^{1L}(u)$ for linear Abelian 2-node quiver with $z = -\frac{2}{5}$ and $k = 5$. Along the imaginary direction in u -plane the 1-loop determinant is not periodic but rather asymptotes to the thimble integral uniformly and identically regardless of the value of real part. Saddle rims on both $-i\infty, i\infty$ ends are shown to lie on two different “asymptotic planes” for $\Im(\mathcal{Z}^{1L})$ giving Z_1, Z_2 respectively, so to prevent the over-counting of thimble contributions in the integration cycle, we should only focus on the irreducible \mathbf{J} -set. For simplicity, the 1-loop determinant is drawn including a schematic graph of thimbles in \mathbf{J} governed by the complex gradient equations for $S_{\text{eff}} = -\ln(\mathcal{Z}^{1L})$. In doing so, we have basically taken advantage of the equivalence of S_{eff} and \mathcal{Z}^{1L} in terms of their saddle-pole structure.

(Here, use was made of the formula (101). For more information, the reader is encouraged to look at its derivation in section [7.2].) The piece-wise FI parameter-dependency of $n_i(\zeta)$ in general is just a direct consequence of the fact that they do not show up inside the trace function of (31) explicitly. It should be stated, however, that this dependency is rather shown to be in a less clear way on the contour of integration γ in the JK residue operation. The Picard-Lefschetz theory does seemingly suggest that this contour is of the form:

$$\gamma(\zeta) = n_1(\zeta)\mathcal{J}_1 + n_2(\zeta)\mathcal{J}_2. \quad (40)$$

Vividly, any jump in the index is related immediately to the jumps in $n_i(\zeta)$ caused by either Stokes phenomena or moving through FI chambers or both. This latter one will be more apparent in the cases where the index in the other FI chambers is non-zero such as generalized XYZ model. In other words, it is possible that applying Lefschetz decomposition may end up with an *on-the-wall* index and any probable wall-crossing phenomenon is lurking here (see section [8]).

This index is obviously invariant under $z \rightarrow -z$ or, equivalently, $y \rightarrow y^{-1}$ exchange symmetry that can be understood in terms of Lefschetz thimbles by noticing that all it does is for the good regions to be flipped around imaginary axis in fig. [5] without thimble integration affected. Because the same pole is again encircled by the thimbles at work here, the JK-residue operation will yield the same result as well.

Next, we should match this result with formula (36) for $\mathcal{M} = \mathbb{P}_{k-1}(\mathbb{C})$ with $D = k - 1$. Let us first simplify things further by noting the Hodge numbers in (36) can be extracted from the Betti numbers in the following way:

$$\begin{aligned} \mathcal{I}(y, \zeta) &= \sum_{p=0}^{k-1} b_p(\zeta) \frac{y^{2p-k+1} - y^{2p-k+3}}{1 - y^2} \\ &= \sum_{p=0}^{k-1} \sum_{q=0}^{k-1} (-1)^{p-q} h_{(p,q)}^{\mathbb{P}_{k-1}(\mathbb{C})} \frac{y^{2p-k+1} - y^{2p-k+3}}{1 - y^2}, \end{aligned} \quad (41)$$

where

$$b_p(\zeta) = \sum_{q=0}^{k-1} (-1)^{p-q} h_{(p,q)}^{\mathbb{P}_{k-1}(\mathbb{C})}(\zeta), \quad (42)$$

are the Betti numbers of the cohomology of $\mathbb{P}_{k-1}(\mathbb{C})$. Notice that for our later purposes, a slight manipulation in the original form of (36) was made to cast it in a more comparable form to the one derived using the Picard-Lefschetz theory in (39a). Equating (41) with the index formula obtained in (39a) imposes the condition

$$b^{\mathbb{P}_{k-1}(\mathbb{C})}(\zeta) = n_1(\zeta) = -n_2(\zeta). \quad (43)$$

where

$$b_0 = b_1 = \dots = b_{k-1} \equiv b^{\mathbb{P}_{k-1}(\mathbb{C})}. \quad (44)$$

This is a very elegant correspondence between Betti numbers of the Dolbeault cohomology groups over the compact moduli space of the supersymmetric vacua in the 2-node quiver and the intersection coefficients of the \mathcal{J} -cycles in the relative homology drawn from the complex Morse function S_{eff} over a non-compact 2-manifold $X = \mathbb{C}^\times$ which is essentially the path space of the quiver path integral. This actually is a *finite* dimensional path space reduced from the original infinite dimensional path space of the $2d \mathcal{N} = (2, 2)$ supersymmetric gauge theory by localization. The power of this correspondence is that even if one does not know the intersection numbers, which happen to be so complicated to obtain in more general cases, the condition (43) determines the relation between n_1, n_2 . Mathematically, $\gamma \subset H_1(\mathfrak{F}; \mathbb{Z})$ where $H_1(\mathfrak{F}; \mathbb{Z})$ is an “ordinary” homology of $\mathfrak{F} \subset \mathbb{C}^\times$ while the constituent cycles still fall into relative homology groups. γ , a closed loop encircling one singularity at $u_* = 0$, gives the betti number $b_1 = 1$ for $H_1(\mathfrak{F}; \mathbb{Z})$. This is the statement that the Poincaré function for the relative homology $H_1(\mathfrak{F}, \mathfrak{F}_{\tau^+}; \mathbb{Z})$ is in fact a Poincaré series because the vector multiplet contribution to 1-loop determinant is $\propto \frac{1}{1-y^2}$. Finally, this correspondence is $H_1(\mathfrak{F}; \mathbb{Z}) \cong H^1(\mathbb{P}_{k-1}(\mathbb{C}); \mathbb{Z})$ regardless of the value of k ⁷.

⁷ Note that, k appears in the exponent of the chiral multiplet contribution in the 1-loop determinant. Hence, modulo the $U(1)$ vector multiplet components that are not related to the saddle points in u -space e.g., eq. (78), flow lines are independent of k .

The isomorphism $H_1(\mathfrak{F}; \mathbb{Z}) \cong H^1(\mathbb{P}_{k-1}(\mathbb{C}); \mathbb{Z})$ needs a little bit of explanation. In the path integral of the supersymmetric gauge theory, the space to be integrated over is initially infinite dimensional. Localization brings it down to a finite dimensional integral over a specific contour, γ , in the space of saddle points of the low-energy theory. These saddles are simply the solutions to supersymmetric BPS conditions that form a moduli space. γ is fixed by Picard-Lefschetz theory to be (40), which is a middle-dimensional cycle- e.g. a smooth 1-manifold in \mathfrak{F} for a quiver system of $\alpha = 2$ mutated by decoupling a $U(1)$ factor. Because the constituent pieces of this contour are the thimbles running all the way to infinity along imaginary directions, γ could be shrunk into a compact submanifold $\mathfrak{W} \subset \mathfrak{F}$ with $\{G_1, G_2\} \cap \mathfrak{W} \neq \emptyset$ ⁸ by way of introducing a regulator. This simply induces a *deformation* retract $\mathfrak{W} \hookrightarrow \mathfrak{F}$ and thus $H_1(\mathfrak{F}; \mathbb{Z}) \cong H_1(\mathfrak{W}; \mathbb{Z})$. The induction by the regulator is not unique in the sense that the path integral, that is, the refined Witten index $\mathcal{I}(y, \zeta)$, does not depend on the choice of regulator as long as gauge symmetry is respected.

To fully compute the Betti numbers b_p , one always needs a generating function that is also difficult to obtain due to the topological complications of the moduli spaces. In simple cases such as $\mathcal{M} = \mathbb{P}_{k-1}(\mathbb{C})$ or Grassmannians, where closed loops are absent in the quiver, this problem has been settled by Reineke in [30]. There, counting supersymmetric ground states of quantum mechanics on the moduli space of a 2-node quiver with k arrows, and dimension vector $(1, n)$ associated with nodes of the quiver is shown to be related to the cohomology of a Grassmannian $\mathcal{M} = \mathbf{Gr}(n, k)$ in an explicit way:

$$P(y) \equiv \sum_{p=0}^{n(k-n)} b_p y^{2p} = \frac{\prod_{j=1}^{k-1} (1 - y^{2j})}{\prod_{j=1}^n (1 - y^{2j}) \prod_{j=1}^{k-n} (1 - y^{2j})}. \quad (45)$$

Now multiplying all sides by y^{-k+1} and putting $n = 1$ yields the index for $\mathbb{P}_{k-1}(\mathbb{C})$ model. From this, we find $b^{\mathbb{P}_{k-1}(\mathbb{C})} = 1$ which was expected from the analysis of Lefschetz thimbles earlier. The $\zeta > 0$ is a geometric phase for $\mathcal{N} = 4$ and thus the $\mathbb{P}_{k-1}(\mathbb{C})$ model has a non-zero index in this phase, whose space of supersymmetric ground states is of dimension

$$h_{(p,q)}^{\mathbb{P}_{k-1}(\mathbb{C})}(\zeta > 0) = \begin{cases} 1 & \text{if } p = q = 0, \dots, k-1 \\ 0 & \text{if otherwise,} \end{cases} \quad (46)$$

[17]. Introducing this in (42) implies correctly the result $b^{\mathbb{P}_{k-1}(\mathbb{C})} = 1$ and affirms the Picard-Lefschetz analysis.

If we were to follow things starting from the Picard-Lefschetz theory side, we would have hit a snag to get the cohomology of the moduli space. The reason for this is

⁸ Certainly, it is not possible to include any of good regions in their entirety in \mathfrak{W} as they vary with respect to y and may grow out of \mathfrak{W} .

straightforward to understand. Consider the expansion of the integration cycle (40) is given. Using stationary phase formula modulo the infinity factor, the values of 1-loop determinant at saddles u_1^*, u_2^* , i.e., Z_1, Z_2 give thimble integrals over $\mathcal{J}_1, \mathcal{J}_2$ (see Subsection [7.1] for explicit calculation). Z_1, Z_2 do not, however, have any finite expansion in y because of the divergence at $y = \pm 1$ caused by the singularities of the vector multiplet contribution but apparently the combination does have a Laurent expansion once n_1 and n_2 are set to be $+1$ and -1 , respectively. It is naturally deemed on general grounds that the dependence on ζ is piece-wise. So the intersection coefficients in the relative homology of \mathcal{J} -cycles should know about the cohomology of the moduli space of supersymmetric quiver vacua.

Still, we need to look for means of generating these coefficients and clarifying what their link to FI parameters is. In the following, we try to put the idea of wall-crossing phenomenon and Stokes jumps into perspective together to address these questions. For a detailed mathematical account of wall-crossing, we giude the reader to consult the beautiful text by Kontsevich and Soibelman [31].

5.2. Regularization of infinite saddles using FI parameters: Jumping off the wall

Even though matching the relative homology of the \mathcal{J} -cycles derived from the Picard-Lefschetz theory with the cohomology of the moduli space of stable quiver representations derived using algebraic topological tools is very instructive as it stands, one may be concerned about the saddle rims at infinity and whether or not studying the Picard-Lefschetz theory is legitimate in such cases due to the degeneracy of saddle points.

As a result we should seek out an on-the-wall interpretation of the index by identifying (40) with the walls of ζ -space itself [17] where a wall encodes data on the BPS configuration inside of the chambers sharing it. If we take this as a fact, given that γ is essentially sitting on a Stokes ray, going off the ray gives rise to Stokes jumps responsible for any possible change in the relative homology of the \mathcal{J} -cycles before hitting the wall from the opposite side:

$$\mathcal{J}_i \rightarrow \mathcal{J}_i + \sigma_{ij} \mathcal{J}_j, \quad n_j \rightarrow n_j + \sum_i \sigma_{ij}, \quad \sigma_{ij} \in \mathbb{Z}. \quad (47)$$

Fix a parameter μ that triggers (47) in a two-chamber theory ($i = 1, 2$). Since we now have a homology cycle, say \mathcal{C}_μ off the Stokes line to the right of it ($\mu \rightarrow 0^+$) or else ($\mu \rightarrow 0^-$), that is locally constant as prescribed by σ_{ij} to be so, Stokes jumps by themselves are not why the index jumps. Stokes jumps are immediately accompanied by variation of FI parameters responsible for building different chambers in the moduli space of BPS states not the u -space. Thus, the index experiences contributions from a perhaps different BPS configuration depending on

how \mathcal{J}_i change. Therefore,

$$\gamma = n_+(\zeta) \mathcal{C}_+ + n_-(\zeta) \mathcal{C}_- \quad (48)$$

where $n_\pm(\zeta) = \frac{1}{2}n_1(\zeta) \pm \frac{1}{2}n_2(\zeta)$ are normalized coefficients that formulate FI chambers. From (39a) we have $n_+ = \Theta(\zeta)$, $n_- = 0$, meaning that as soon as $\mu = 0^+$,

$$\gamma = \Theta(\zeta) \mathcal{C}_+ \equiv \Theta(\zeta_2) \mathcal{C}_+ + 0 \mathcal{C}_-. \quad (49)$$

The jumps through the Stokes ray are hence the reason for variations of contours $\mathcal{C}_- \leftrightarrow \mathcal{C}_+$ in a (locally) homologically invariant way so the index jump is not a result of Stokes jumps alone. The homology cycles \mathcal{C}_\pm are associated with two chambers defined by the conditions that coefficients $n_\pm(\zeta)$ hold onto. Nonetheless, one still needs to clarify what μ is and how it is related to FI parameters.

There is a common escape route that allows the thimble analysis to be done in a finite region more systematically that could explain the μ -dependence of n_i explicitly with the help of μ . Assuming that γ is on a Stokes ray, one can add an arbitrary smooth gauge-invariant regulator of the form $\mu(\zeta)R(u)$ to (37), where $R(u)$ vanishes at the poles of \mathcal{Z}^{1L} in X and $\mu(\zeta) = 0$ characterizes the wall in ζ -space, forcing simultaneously the condition for flow lines to lie on a Stokes ray in the u -space. Giving up gauge-invariance in $X \setminus \mathfrak{F}$, we can define the monomial

$$R(u) = \prod_{i=1} (u - u_i^P)^2 \quad (50)$$

where u_i^P are poles in \mathfrak{F} . Thus, $\mu(\zeta)$ is a regularization parameter for the u variables that allows the effective action to have its non-isolated degenerate saddle points re-incarcerated as isolated nondegenerate ones within a finite region. On Picard-Lefschetz theory grounds [6], it means that the *regularized* complex Morse function $h_{\mathbb{C}} := -\ln \mathcal{Z}_{\text{reg}}^{\text{1L}}$ has finitely many isolated nondegenerate critical points of Morse index 1, so it is a perfect Morse function. Then, the rank of $H_1(\mathfrak{F}, \mathfrak{F}_{\tau^*}; \mathbb{Z})$ for even a non-compact \mathfrak{F} is exactly the number of critical points of $h_{\mathbb{C}}$ in \mathfrak{F} , equal to the number of \mathcal{J} -cycles in \mathbf{J} -set.

For a $(0+1)d$ $U(1)$ supersymmetric quantum gauge theory, the refined index then becomes

$$\begin{aligned} \mathcal{I}(y) &= \kappa^{-1} \int_{\mathcal{C}_{\mu(\zeta)}} e^{\mu(\zeta)R(u) - S_{\text{eff}}(u)} du \\ &\equiv -\kappa^{-1} \sum_i \oint_{u=u_i^P} e^{-S_{\text{eff}}(u)} du \end{aligned}$$

where $\mathcal{C}_{\mu(\zeta)}$ is a closed μ -deformed integration cycle that encircles the poles u_i^P . Note that $\mathcal{C}_0 \equiv \gamma = \sum_i n_i \mathcal{J}_i$. Locally speaking, there is no obstruction in the moduli space of the μ -deformed theory other than those of S_{eff} , so $\mathcal{C}_\mu(\zeta)$ retracts to γ smoothly and n_i should hence be fixed in a way that \mathcal{C}_0 is still a closed loop. We will discuss two methods by which we could determine the coefficients n_i in some simple cases without using a regulator in the next subsection.

What if we have more than two chambers? Then μ should be analytically continued to complex plane i.e. $\mu(\zeta)e^{i\theta}$ and by changing θ the cycle $\mathcal{C}_{\mu(\zeta)e^{i\theta}}$ will be rotated such that the integration cycle on the Stokes ray, γ , is approached by $\mathcal{C}_{0e^{i\theta}}$. The point 0 is the bifurcation or junction point of all the segments of the wall and phase factor $e^{i\theta}$ allows a variation of the integration cycle around this point that fall into different chambers in the corresponding ζ -space. Therefore, we can take θ as describing a single point in the ζ -space of FI parameters.

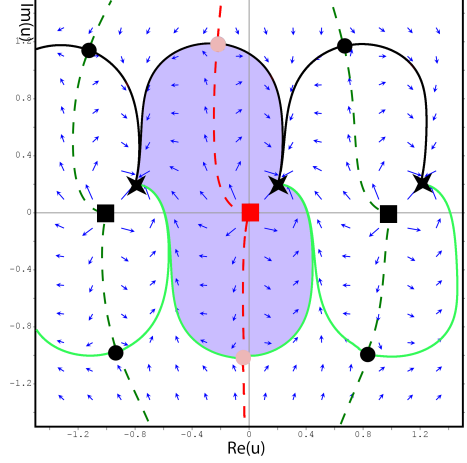


Figure 7. The path space of regularized 2-node quiver model. Here, the regulator is $\frac{\mu}{2}u^2$ with $\mu = 0.01$ and $z = 0.2(1+i)$ so R -symmetry is anomalous. The total gauge group is $G = U(1)^2$ and the FI term is $\zeta(D) = \zeta_1 D_1 + \zeta_2 D_2$. The condition $\zeta_1 + \zeta_2 = 0$ decouples one $U(1)$ and amputates one node. We take $\mu = \zeta_2 = -\zeta_1$. The regularization term is not invariant under $u \rightarrow u + 1$ so the regularized theory should only be studied in the purple domain whose boundary is a great semi-circle on the compactified subspace -a sphere. We want to emphasize that partial compactification comes as a result of truncating the \mathcal{K} -cycles. Because of this regularization, Stokes lines are gone and \mathcal{J} -cycles are confined to form an ordinary homology, replacing the relative homology, associated with the invariant quantity, namely the non-vanishing index in the chamber $\zeta_2 > 0$. Rank of $H_1(\mathfrak{F}, \mathfrak{F}_{\tau^*}; \mathbb{Z}) = 2$.

In the context of $\mathcal{N} = 1$ supersymmetric theories in four dimensions with multiple discrete vacua disconnected by $1/2$ -supersymmetric domain walls, authors of [32, 33] have shown that these domain walls can meet at $1d$ junctions which preserve $1/4$ of the supersymmetry of the Hamiltonian. These junctions may equally be interpreted as $1/2$ -supersymmetric defects on the interface of the domain walls involved. In the Picard-Lefschetz theory, such junctions happen to coincide with the Stokes lines because of the existence of 3-junction joints, which would be a perfect example to study for homological purposes in higher dimensional gauge theories.

5.3. Intersection numbers in quiver quantum mechanics

Witten index check. This method is based on the fact that a correct integration cycle γ has to reproduce the Witten index (which is the Euler characteristic of the target manifold in certain cases such as $\mathbb{P}_{k-1}(\mathbb{C})$ and $\mathbf{Gr}(n, k)$ theories), as a limiting case of the refined index once all fugacities for the Cartan directions of the gauge group are turned off [34]. Also, the index formula we are advocating in this work still does not seem to give a solid relation between the FI parameters and intersection numbers mostly and little is known about the relative homology techniques that one may in practice employ to compute these numbers, which in principle determine the degeneracies of BPS states (see footnote [1]). In this regard, the theory of spectral networks have been successful in determining these degeneracies and the geometry governing them in $4d \mathcal{N} = 2$ theories [35].

The Picard-Lefschetz theory prescribes for general thimble configurations, the following abstract formula:

$$n_i = \langle \gamma | \mathcal{K}_i \rangle, \quad (51)$$

which, including the sign, actually counts the number of \mathcal{K} -cycles going through a specific pole encircled in γ and crossing γ at i th \mathcal{J} -cycle. Since the flow equations are highly non-linear in the examples we study, this formula is of little practical use for $\alpha > 1$ as plotting Lefschetz thimbles of real codimension ≥ 2 turns out to be beyond hope. So in general we do not have a practical tool for understanding the Morse flows and so on.

The situation, however, is not as bad as it sounds. For example, we might guess the coefficients n_i by looking at the Witten index. For this, we go back to the formula (39a) and check if the formula is finite once $y \rightarrow 1$ or, equivalently, $z \rightarrow 0$. This gives

$$\mathcal{I}(\zeta) \xrightarrow{y \rightarrow 1} \frac{1}{0} (n_1(\zeta) + n_2(\zeta)). \quad (52)$$

So upon choosing $n_1(\zeta) = -n_2(\zeta)$, we get a speculative Witten index, finite yet undetermined. Similarly, for a generic Witten index, the constraint

$$\sum_{i=1}^{2^\alpha} n_i(\zeta) = 0, \quad (53)$$

has to be fulfilled for finiteness. Moreover, for \mathcal{M} with a Kähler geometry of complex dimension D , the Witten index derived from (36) is defined in the geometric phase as [2]

$$\mathcal{I} = \sum_{p=0}^D b_p = \chi(\mathcal{M}). \quad (54)$$

This already puts a strong constraint on what the coefficients are. Since $n_i = \pm 1$ for a convergent thimble

integral, our formula in the geometric phase gives

$$\chi(\mathcal{M}) = \lim_{y \rightarrow 1} \frac{1}{\kappa^\alpha} \sum_{j=1}^{2^\alpha} n_j Z_j(y) \quad (55)$$

The right-hand side must have a finite limit so the coefficients n_j only take certain values good for this deed. Given that we do not have prior knowledge of the Witten index for some \mathcal{M} , n_j have to be unique up to an overall sign for the index $\chi(\mathcal{M})$. We propose that the overall sign should be consistent with that of each thimble integral once written as a series in y . For example, the constraint (53) forces $n_2 = -n_1$ for the $\mathcal{N} = 4$ decoupling Abelian 2-node quiver with k -bifundamentals and $\zeta > 0$, so

$$\chi(\mathbb{P}_{k-1}(\mathbb{C})) = n_1 k. \quad (56)$$

Taking into account the overall sign of each Lefschetz thimble integrals ($= y^{-k+1} \sum_{i=0}^{\infty} y^{2i}$ and $= y^{k+1} \sum_{i=0}^{\infty} y^{2i}$), namely, $+1$, we conclude that $n_1 = 1$.

The downside of this check is that the divergence in the vector multiplet contribution to 1-loop determinant simply does not disappear from the index. Alternatively, the vector multiplet determinant is derived from a twisted chiral multiplet of axial R -charge 2 and vanishing vector R -charge. Chiral multiplet of this $(2, 2)$ theory, i.e., $\Sigma(\sigma, \lambda, F_{12} + iD)$ in the adjoint representation, entails an auxiliary scalar field D that comes with the field strength in a linear form. Taking the χ_y genus limit, $\text{tr} \Sigma^2$ is proportional to [16]

$$\frac{y}{1 + y^2} \quad (57)$$

which is divergent at $y^2 = -1$. For $\mathcal{N} = (2, 2)$ $SU(2)$ gauge theory with k fundamental chiral multiplets, this contribution to the refined Witten index will cause the index to be singular when k being even. So Witten index check fails to give the intersection numbers.

Alternative check. If we go back to (51), we notice that \mathcal{K} -cycles have their boundaries lie at the singularities of the u -space X , which combined with the fact that $\langle \mathcal{K}_i, \mathcal{J}_j \rangle = \delta_{ij}$ provides the best clue on how to obtain the intersection coefficients. So tracking the \mathcal{K} -cycles ending at the only singularity of $\mathfrak{F} \subset X$, made it possible in the simplest case of a 2-node quiver to get $n_1 = -n_2 = +1$:

$$\langle \gamma, \mathcal{K}_1 \rangle = +1, \quad \langle \gamma, \mathcal{K}_2 \rangle = -1.$$

As $x = e^{\kappa u}$, this singularity at $u = 0 \pmod{\mathbb{Z}}$ due to $u \sim u + 1$ becomes the point $x = 1$ in the x -parametrization of u -space, that will be used in what follows. The saddle rims at $\Im(u) = \pm\infty$, or similarly $|x| = 0, \infty$, together with the only singularity point divide \mathfrak{F} into two regions D_1 and D_2 such that $\mathcal{K}_{1,2} \subset D_{1,2}$ and $x = 1 \in D_1 \cap D_2$. The main conclusion to draw was that $\text{sign}(n_1) = -\text{sign}(n_2)$. Same strategy will be generalized to higher order (non)Abelian quivers with rank-one gauge groups.

Let G be the total gauge group of the gauged quantum mechanics with rank α . We will also assume that G is a

product of unitary groups with bifundamental representations as well as at least one $U(1)$ gauge group sitting at one of the nodes in an N -node quiver. If every node is labeled by 1, it means the total gauge group is $G = U(1)^\alpha$ where $N = \alpha$. In 4d $SU(M)$ SYM, these N nodes all together with k_{N-1} bifundamental fields determine the bound states of a collection of $N \leq M$ distinct dyons. Here, we will only consider the quivers with rank-one gauge groups at all nodes. The following approach is a thimble-based geometrization of the quiver mutation introduced in [27] and used in [20] to compute refined Witten index.

We first decouple a $U(1)$ factor of the gauge group, sitting at the l th node where there are a total of α nodes. Now we have a linear chain of nodes exactly like $\rightarrow \bigcirc \rightarrow \dots$ with $\alpha - 1$ nodes. Pick out, next, another node, say the $l + \tilde{l}$ th, corresponding to a flavor holonomy \mathbf{x} such that the total number of remaining distinct (bi)fundamental chiral multiplets (arrows) on the mutated quiver is at least $\alpha - 1$. For instance, in closed quivers, the $l + \tilde{l}$ th node might be the one in the immediate neighborhood of the l th. On the contrary, in linear nodes, this might be the last node appearing on the reduced quiver. Then we calculate the *incomplete thimble* functions

$$Z_\infty(y, x_i) := \lim_{\mathbf{x} \rightarrow \infty} \mathcal{Z}^{1L},$$

$$Z_0(y, x_i) := \lim_{\mathbf{x} \rightarrow 0} \mathcal{Z}^{1L}.$$

where i runs over $1, \dots, \widehat{l}, \dots, \widehat{l + \tilde{l}}, \dots, \alpha^9$ with \widehat{l} meaning that l is omitted. In a, now, $\alpha - 2$ dimensional path space X of flavor holonomies parametrized by $x_i = e^{\kappa u_i}$, these functions determine a couple of $\alpha - 2$ complex dimensional unbounded manifolds of co-dimension 2 defined by

$$F_{\infty,0}(x_i, y) = Z_{\infty,0}(x_i, y) \hookrightarrow \mathbb{C}^{\alpha-2}.$$

Let us call $\tilde{F}(y, x_i)$ the restriction of F to the submanifolds parametrized by only real part of x_i, y . We now consider the maps

$$\beta_j : \mathcal{J}_j \rightarrow \tilde{F}_\infty, \quad \beta_k : \mathcal{J}_k \rightarrow \tilde{F}_0, \quad (58)$$

with $j \in J$ and $k \in K$, where β_j, β_k are inclusion maps by restriction to real variables $x_i \in \mathbb{R}_+^{\alpha-2}$ and $0 < y \leq 1$. The result of thimble integrals are now just a real homologically invariant function of y , $\tilde{Z}_{j,k}(y) = \int_{\mathcal{J}_{j,k}} \mathcal{Z}^{1L}(x_i, y) \prod_{a \neq l, l + \tilde{l}}^\alpha dx_a$. We also bear in mind that the spectrum of Ramond sector is invariant under charge conjugation and therefore $\mathcal{I}(\zeta, y^{-1}) = \mathcal{I}(\zeta, y)$. In our current treatment, this simply means that

$$y \longleftrightarrow y^{-1} \iff \tilde{F}_0 \longleftrightarrow \tilde{F}_\infty. \quad (59)$$

Now we assume that $\text{sign}(n_j) = -\text{sign}(n_k)$ for any $x_i \in \mathbb{R}_+^{\alpha-1} \setminus \mathcal{S}$ where $\mathcal{S} = \{\text{singularities}\}^{10}$. Then, in the context of quiver systems with the aforementioned qualities, the following conjecture holds good:

Conjecture: *\mathcal{J} -cycles along the Stokes rays either on \tilde{F}_∞ or \tilde{F}_0 will have opposite intersection numbers as all the singularity curves, surfaces or hypersurfaces are crossed. Using (59), we have that if $\text{rank}(G) = \alpha$ is odd, then \mathcal{J} -cycles along the Stokes rays on \tilde{F}_∞ and \tilde{F}_0 will have opposite intersection numbers against one another, and the same for even α regardless of the singularity content of \mathcal{S} .*

The remaining $2^{\alpha-1} - |J| - |K|$ of intersection coefficients are immediately zero. Here, $|J|$ is the cardinality of the index set J .

We have all pieces needed to determine the index modulo an overall sign. The refined Witten index in the presence of R -symmetry is given by

$$\kappa^{\alpha-1} \mathcal{I}(y, \zeta) = \sum_{j=1}^{|J|} Z_j(y, \zeta) - \sum_{k=1}^{|K|} Z_k(y, \zeta), \quad (60)$$

where ζ dependency has been integrated with the thimble integrals. For overall sign, we will stick with the same proposal prescribed below (55). As a quick consistency check, we can see that for a 2-node quiver, this formula correctly produces $\mathcal{I}(y, \zeta) = \kappa^{-1}(Z_1 - Z_2) = y^{-k+1} \sum_{i=0}^{k-1} y^{2i}$ for $\zeta > 0$.

We conjecture that if $|K| = |J|$, we can write the index formula (60) for a quiver theory with gauge group $G \cong U(1)^\alpha$ and an overall $U(1)$ factor decoupled as

$$\mathcal{I}(y, \zeta) = \kappa^{1-\alpha} \sum_{i=1}^{2^{\alpha-1}} (-1)^{t_{i-1}} Z_i(y, \zeta) \quad (61)$$

where t_i is the Thue-Morse sequence $t_{i \geq 0} = 0110100110010\dots$, and taken to be as the generating function for the intersection coefficients n_i in the theory. Here, the order of y in the numerator of Z_i is nondecreasing with increasing i . Note that the magnitude of intersection coefficients $(-1)^{t_{i-1}}$ is one, being in accord with the coefficients of expansion of a stable BPS bound state of the quiver theory in terms of elementary BPS states [27].

3-node quiver

Let us consider the 3-node linear quiver with k_i bifundamental chiral multiplets between the i th and the $i + 1$ th nodes for $i = 1, 2$. We decouple a $U(1)$ from the quiver

⁹ The case $\alpha = 2$ is straightforwardly checked by plotting the flow lines as we did for the $\mathbb{P}_{k-1}(\mathbb{C})$ model.

¹⁰ For instance, in quiver theories with trivial superpotential, \mathcal{S} only contains one element, which is exactly at $x_N = 1$ for a quiver with N nodes.

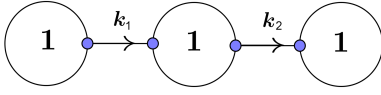


Figure 8. A generalization of $\mathbb{P}_{k-1}(\mathbb{C})$ model. These are 3 nodes that all together form a linear quiver with rank $\alpha = 1$ Abelian gauge groups at each node. k_i , the number of arrows, also shows the number of bifundamentals.

in a way that the 3-node $U(1)$ quiver model, the 1-loop determinant is given by

$$\mathcal{Z}^{1L} = \frac{\kappa^2 y^{-k_1-k_2+2} \left(\frac{y^2-x_2}{1-x_2} \right)^{k_1} \left(\frac{x_2 y^2-x_3}{x_2-x_3} \right)^{k_2}}{(1-y^2)^2}. \quad (62)$$

The thimble integrals are given by

$$Z_1(y) = \lim_{\substack{x_2 \rightarrow \infty \\ x_3 \rightarrow \infty}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^{-k_1-k_2+2}}{(1-y^2)^2},$$

$$Z_2(y) = \lim_{\substack{x_2 \rightarrow \infty \\ x_3 \rightarrow 0}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^{-k_1+k_2+2}}{(1-y^2)^2},$$

$$Z_3(y) = \lim_{\substack{x_2 \rightarrow 0 \\ x_3 \rightarrow \infty}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^{k_1-k_2+2}}{(1-y^2)^2},$$

$$Z_4(y) = \lim_{\substack{x_2 \rightarrow 0 \\ x_3 \rightarrow 0}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^{k_1+k_2+2}}{(1-y^2)^2}.$$

Now using the formulae

$$\gamma(\zeta) = \sum_{i=1}^4 n_i(\zeta) \mathcal{J}_i(y), \quad (63a)$$

$$\mathcal{I}(y, \zeta) = \kappa^{-2} \sum_{i=1}^4 n_i(\zeta) Z_i(y). \quad (63b)$$

Witten index constraint forces

$$\begin{aligned} \mathcal{I}(y, \zeta) &= \frac{-(n_2 + n_3 + n_4) y^{-k_1-k_2+2} + n_2 y^{-k_1+k_2+2}}{(1-y^2)^2} \\ &+ \frac{n_3 y^{k_1-k_2+2} + n_4 y^{k_1+k_2+2}}{(1-y^2)^2} \\ &= \frac{y^{-k_1-k_2+2}}{(1-y^2)^2} (-n_2 - n_3 - n_4 + n_2 y^{2k_2} \\ &+ n_3 y^{2k_1} + n_4 y^{2k_1+2k_2}), \end{aligned} \quad (64)$$

where we have dropped the ζ from the coefficients for simplicity. For a finite Witten index, the numerator must divide the denominator. The expression in the parenthesis can be factorized as $(a + n_3 y^{2k_1})(b + n_2 y^{2k_2})$ where

$$-ab = n_2 + n_3 + n_4, \quad n_3 n_2 = n_4, \quad a n_2 = n_2, \quad b n_3 = n_3, \quad (65)$$

which in the non-vanishing chamber, immediately result in

$$-1 = n_2 + n_3 + n_3 n_2.$$

Finally, under $k_1 \leftrightarrow k_2$ exchange, the index (64) remains unchanged so $n_2 = n_3$. Therefore, $n_2 = n_3 = -1$ and $n_1 = n_4 = 1$, giving the correct nonzero index with the overall plus sign

$$\mathcal{I}(y) = \left(y^{-k_1+1} \sum_{i=0}^{k_1-1} y^{2i} \right) \left(y^{-k_2+1} \sum_{i=0}^{k_2-1} y^{2i} \right). \quad (66)$$

One may check straightforwardly that the formula (61) exactly produces this result that is instantly generalizable to all longer Abelian linear N -node quivers ($N > 3$).

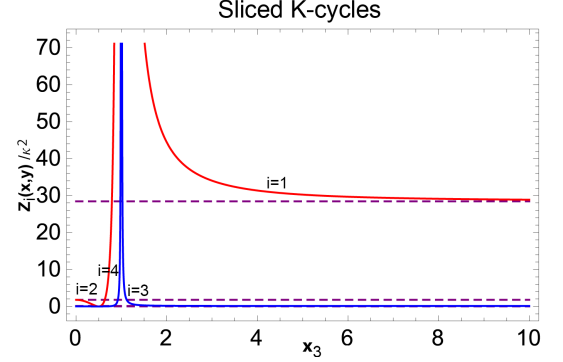


Figure 9. All four possible contributions of thimble integrals in the refined index of 3-node quiver with $y = 1/2$. Here, the existence of two vertical cusps results in $n_1 = -n_2 = +1$, $n_3 = -n_4 = -1$. The jump discontinuity takes place at $x_3 = 1$.

6. ORIENTED CLOSED QUIVERS

6.1. XYZ model

The triangular quiver with the superpotential $W = (XYZ)^k$ is depicted in fig. [10] for a positive integer number k . The R -charges satisfy

$$R_X + R_Y + R_Z = \frac{2}{k}. \quad (67)$$

Decoupling the first $U(1)$, the 1-loop determinant is given as

$$\mathcal{Z}^{1L} = \frac{-\kappa^2 y^{-1} (y^2 - x_2 y^{R_X}) (y^{R_Z} - x_3 y^2) (x_2 y - x_3 y^{R_Y})}{(1-y^2)^2 (1-x_2 y^{R_X}) (x_3 - y^{R_Z}) (x_2 - x_3 y^{R_Y})} \quad (68)$$

Calculating the thimble integrals, one finds

$$Z_1(y) = \lim_{\substack{x_2 \rightarrow \infty \\ x_3 \rightarrow \infty}} \mathcal{Z}^{1L} = \kappa^2 \frac{y}{(1-y^2)^2},$$

$$Z_2(y) = \lim_{\substack{x_2 \rightarrow 0 \\ x_3 \rightarrow 0}} \mathcal{Z}^{1L} = \kappa^2 \frac{y}{(1-y^2)^2},$$

$$Z_3(y) = \lim_{\substack{x_2 \rightarrow \infty \\ x_3 \rightarrow \infty}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^3}{(1-y^2)^2},$$

$$Z_4(y) = \lim_{\substack{x_2 \rightarrow \infty \\ x_3 \rightarrow 0}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^3}{(1-y^2)^2}.$$

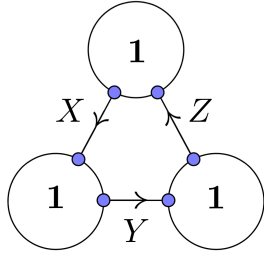


Figure 10. Triangular quiver has three rank-one gauge groups at each vertex and the bifundamentals are X, Y, Z .

The Witten index check in determining the intersection coefficients in this case is doomed as the degree of polynomials in the denominators of Z_i 's is higher than that of the polynomials in the numerators. Instead, for XYZ model and in general cases, we employ the alternative method and verify that the formula (60) yields the correct index only by imposing $k = 1$ in (67).

Because in this model a $U(1)$ factor of the gauge group is decoupled, so as shown in fig. [11] there are a total of four \mathcal{J} -cycles, $\mathcal{J}_1, \mathcal{J}_4$ coming with $n = +1$ and $\mathcal{J}_2, \mathcal{J}_3$ with $n = -1$. We already know $\mathcal{J}_k \in F_\infty$ and $\mathcal{J}_j \in F_0$. The singularity ray described by $x_3^2 = y^{R_Z}$ will not allow $\mathcal{J}_2, \mathcal{J}_3$ to fall in the same homology class and so will be the case with $\mathcal{J}_1, \mathcal{J}_4$. Putting a $U(1)$ aside, we have $2^2 = 4$ thimbles in the irreducible \mathbf{J} -set.

The integration cycle is given by

$$\gamma = \mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 + \mathcal{J}_4. \quad (69)$$

This result is consistent with formula (4.26) of [20] for $k = 1$:

$$\begin{aligned} \mathcal{I}(y) &= \lim_{k \rightarrow 1} \left(\frac{y}{1 - y^{\frac{2}{k}}} - \frac{y^{\frac{2}{k}-1}}{1 - y^{\frac{2}{k}}} \right) \\ &= \lim_{k \rightarrow 1} \left(\frac{y - y^{\frac{2}{k}+1}}{(1 - y^{\frac{2}{k}})^2} - \frac{y - y^{\frac{2}{k}+1}}{(1 - y^{\frac{2}{k}})^2} \right) \\ &= \frac{y}{(1 - y^2)^2} - \frac{y}{(1 - y^2)^2} - \frac{y^3}{(1 - y^2)^2} + \frac{y^3}{(1 - y^2)^2} \\ &= \kappa^{-2}(Z_1 - Z_2 - Z_3 + Z_4) = 0. \end{aligned} \quad (70)$$

This verifies the authors' hypotheses about the genericity and quasi-homogeneity of superpotential dictating the relationship between $U(1)_R$ charges of chiral fields in models where holomorphic gauge invariant monomials are turned on in chiral fields.

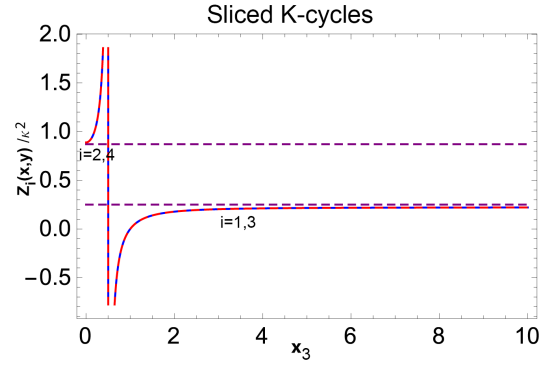


Figure 11. All four possible contributions of thimble integrals in the refined index of XYZ model in the R -charge configuration $(0, 0, 2)$ with $y = 1/2$. Here, the conjecture gives $n_1 = -n_2 = +1, n_3 = -n_4 = -1$. The jump discontinuity occurs at $x_3 = 1/2$.

6.2. $4d \mathcal{N} = 2$ $SU(3)$ Yang-Mills

Finally, we want to examine a non-trivial example concerning the BPS states of $4d \mathcal{N} = 2$ $SU(3)$ Yang-Mills theory with rank-one quiver gauge groups. The BPS particle is a W -boson which is stable and ground states of the quiver quantum mechanics exist at weak coupling regime of the theory. Lefschetz thimble decomposition of γ is found to be

$$\gamma = \sum_{i=1}^8 (-1)^{t_i-1} \mathcal{J}_i. \quad (71)$$

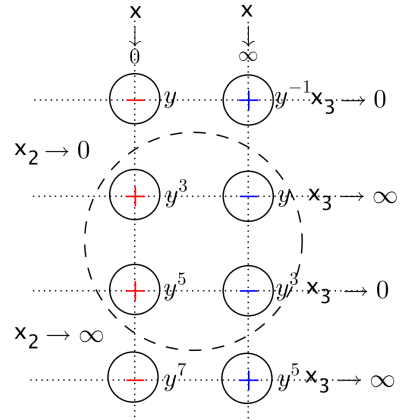


Figure 12. Intersection coefficients in $4d \mathcal{N} = 2$ $SU(3)$ SYM for all Lefschetz thimbles in \mathbf{J} -set. The duality $y \rightarrow y^{-1}$ amounts to a rotation of coefficients around \mathbf{x} -axis by 180 degrees. Self-dual saddle contributions $\propto \pm y^3$ will cancel out because rank of the mutated quiver is 3. As expected, the coefficients in each row have opposite signs (along Stokes curves parallel to x_3 axis).

Refined Witten index of the theory in the non-vanishing

chamber is then

$$\begin{aligned}
\mathcal{I}(y) &= \kappa^{-3} \sum_{i=1}^8 (-1)^{t_{i-1}} Z_i \\
&= \kappa^{-3} (Z_1 - Z_2 - Z_3 + Z_4 - Z_5 + Z_6 + Z_7 - Z_8) \\
&= \frac{y^{-1} - 2y + y^3 - y^3 + 2y^5 - y^7}{(1 - y^2)^3} \\
&= y + y^{-1}.
\end{aligned} \tag{72}$$

There are four chambers in this theory in two of which the index is given by (72) and zero in the other two. In the regularized scheme, there is a 4-point junction point in u -space around which θ changes so we sweep all possible chambers in the ζ -space:

$$\gamma = 0 \mathcal{C}_{0e^{i\theta_1}} + \tilde{n}_2(\zeta) \mathcal{C}_{0e^{i\theta_2}} + 0 \mathcal{C}_{0e^{i\theta_3}} + \tilde{n}_4(\zeta) \mathcal{C}_{0e^{i\theta_4}} \tag{73}$$

where $\theta_4 > \theta_3 > \theta_2 > \theta_1$. $\tilde{n}_2(\zeta)$ and $\tilde{n}_4(\zeta)$ depend on equations of the corresponding chambers in terms of $\Theta(\zeta)$ and are normalized to 1.

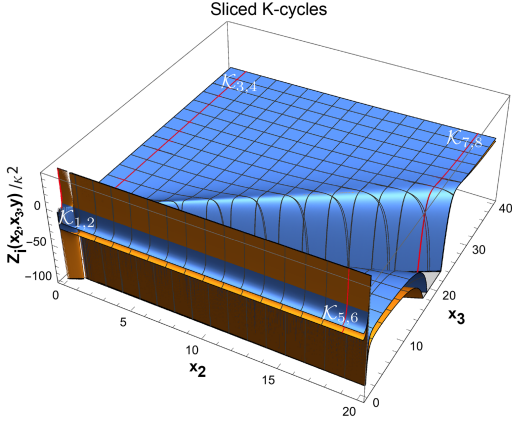


Figure 13. The number of \mathcal{K} -cycles shows the total number of cycles forming the integration cycle γ . All eight possible contributions of thimble integrals in the refined index of the quiver for $4d \mathcal{N} = 2$ $SU(3)$ Yang-Mills.

7. MORE ON $\mathcal{N} = 4 \mathbb{P}_{k-1}(\mathbb{C})$ MODEL

7.1. Explicit calculation of thimble integrals

Instead of solving flow equations which are highly non-linear, we focus on the alternative definition of Lefschetz thimbles using the conservation of Hamiltonian flow governed by the effective action $S_{\text{eff}}(u, y)$ along the thimble \mathcal{J}_i , namely

$$\Im(S_{\text{eff}}(u, y)) = \Im(S_{\text{eff}}(u_i^*, y)) = \text{const.} \tag{74}$$

where $u_i^* \in \mathcal{J}_i$. Note that the auxiliary field D has already been integrated out so that the effective action does not contain the FI parameter, being now carried over into

the intersection coefficients $n_i(\zeta)$ (see subsection [7.2]). So wall-crossing phenomena will be independent of the thimbles themselves but will arise rather cleverly in the way these thimbles cross the integration contour γ .

For our example of an Abelian linear 2-node quiver with k bifundamentals and gauge fields of rank-one at each node, one has

$$\begin{aligned}
S_{\text{eff}}(x, y) &:= -\ln(\mathcal{Z}^{1\text{L}}(x, y)) \\
&= -\ln(\kappa y^{1-k} (1 - y^2)^{-1} (x - y^2)^k (x - 1)^{-k}),
\end{aligned} \tag{75}$$

where $x = e^{\kappa u}$ is a parameterization of the holonomy of the gauge field in the Cartan subalgebra \mathfrak{h} of the gauge group G . We want to calculate the integrals

$$Z_{1,2}(y) = \kappa^{-1} \int_{\mathcal{J}_{1,2}} \mathcal{Z}^{1\text{L}}(e^{\kappa u}, y) \frac{dx}{x}. \tag{76}$$

Starting first with \mathcal{J}_1 , we find that this thimble is described by

$$\begin{aligned}
\{x \in \mathfrak{F} \subset \mathbb{C}^\times \mid \Im(\ln(\kappa y^{1-k} (1 - y^2)^{-1} (x - y^2)^k (x - 1)^{-k})) \\
= \Im(\ln(\kappa y^{1-k} (1 - y^2)^{-1}))\}.
\end{aligned} \tag{77}$$

Here, we assume that \mathcal{J}_1 is attached to the $-i\infty$ saddle rim, \mathfrak{B}_- in u -plane. Now the condition for (77) can be written as

$$\Im(\ln((x - y^2)^k (x - 1)^{-k})) = 0. \tag{78}$$

or equivalently,

$$\Im(\ln(c e^{2\pi i m})) = 0 \tag{79}$$

for some $c \in \mathbb{R}^+$ and $m \in \mathbb{Z}$. This is because the effective action S_{eff} is not single-valued but we will restrict the logarithm to the principal branch $m = 0$.

Let us for the moment take the $U(1)_R$ holonomy parameter $z \in \mathbb{R}$, meaning the (left-moving) R -symmetry on T^2 is left to be anomalous. On the other hand, we can exploit the fact that $\bar{y} = y^{-1}$ to put (78) into the following form:

$$y^{2k} \frac{(x - y^2)^k (\bar{x} - 1)^k}{(y^2 \bar{x} - 1)^k (x - 1)^k} = 1, \tag{80}$$

meaning that the expression inside logarithm function has to be real. Positivity should be considered separately only if it does not follow the reality condition on the solution immediately.

A solution of (80) is written as

$$\mathcal{J}_1 : \begin{cases} -\infty \leq \tau < -\tau_B, & \Re(u) = z, -\infty < \Im(u) \leq 0 \\ -\tau_B \leq \tau \leq \tau_B, & z \leq \Re(u) \leq z + 1, \Im(u) = -\infty \\ \tau_B < \tau \leq \tau^*, & \Re(u) = z + 1, -\infty < \Im(u) \leq 0 \end{cases} \tag{81}$$

leading to $c = 1$. The time $-\tau_B < 0 < \tau_B \ll \tau^*$ marks the beginning of \mathcal{J}_1 flowing on the boundary of the space X .

We can show this piece of the flow line more appropriately by $x = \infty e^{\kappa \Re(u)}$ and $z \leq \Re(u) \leq z+1$ that does precisely correspond to the “big-circle” boundary of the u -space in x -parametrization illustrated as a contributing element of the simple wall-crossing formula of [17].

Similarly, \mathcal{J}_2 is described by

$$\mathcal{J}_2 = \{x \in \mathfrak{F} \subset \mathbb{C}^\times | \Im(\ln(\kappa y^{1+k}(1-y^2)^{-1} \times (xy^{-2}-1)^k(x-1)^{-k})) = \Im(\ln(\kappa y^{1+k}(1-y^2)^{-1}))\}, \quad (82)$$

giving the condition

$$y^{-2k} \frac{(xy^{-2}-1)^k(\bar{x}-1)^k}{(\bar{x}-y^{-2})^k(x-1)^k} = 1, \quad (83)$$

which accepts the solution

$$\mathcal{J}_2 : \begin{cases} -\infty \leq \tau < -\tau_B & : \Re(u) = z, 0 \leq \Im(u) < \infty \\ -\tau_B \leq \tau \leq \tau_B & : z \leq \Re(u) \leq z+1, \Im(u) = \infty \\ \tau_B < \tau \leq \tau^* & : \Re(u) = z+1, 0 \leq \Im(u) < \infty \end{cases} \quad (84)$$

that yields $c = 1$. The time $-\tau_B < 0 < \tau_B \ll \tau^*$ marks the start of \mathcal{J}_2 on the other boundary of the space X . This piece of the flow line may be shown by $x = 0e^{\kappa \Re(u)}$ together with $z \leq \Re(u) \leq z+1$ that does precisely correspond to the “small-circle” boundary of the u -space in x -parametrization illustrated as a second element contributing to the simple wall-crossing formula of [17].

The associated \mathcal{K} -cycles are given as

$$\begin{aligned} \mathcal{K}_1 &= \{\Im(u(-\infty)) = 0, \Im(u(\tau^*)) = -\infty, \Re(u(\tau)) = 0\} \\ \mathcal{K}_2 &= \{\Im(u(-\infty)) = 0, \Im(u(\tau^*)) = +\infty, \Re(u(\tau)) = 0\} \end{aligned}$$

where τ changes from $-\infty$ to τ^* .

Both \mathcal{J}_1 and \mathcal{J}_2 are codimension 1 submanifolds of \mathbb{C}^\times whose points can be reached at any τ by a flow that starts at $u = z$ at $\tau = -\infty$. The flow line $u(\tau)$ that reaches $u = z+1$ at time τ^* is accompanied by another flow line $u(\tau - \tau^*)$ that arrives at $u = z+1$ at $\tau = 0$.

On \mathcal{J}_1 , (76) can be re-written as

$$\begin{aligned} \tilde{Z}_1(y) &= \int_{\mathcal{J}_1} \frac{(x' - y^2 + 1)^k}{(x' + 1)x'^k} dx' = \int_{\mathcal{J}_1} \frac{(1 + \frac{1-y^2}{x'})^k}{x' + 1} dx' \\ &= \sum_{n=0}^{\infty} \binom{k}{n} (1-y^2)^n \int_{\mathcal{J}_1} \frac{dx}{(x-1)^{n+1}} \\ &= \kappa \sum_{n=0}^{\infty} \binom{k}{n} (1-y^2)^n \int_{\mathcal{J}_1} \frac{du}{(e^{\kappa u} - 1)^{n+1}}, \quad (86) \end{aligned}$$

where $\tilde{Z}_1(y) = y^{k-1}(1-y^2)Z_1(y)$ and use was made of the binomial expansion since obviously $|x'| = |x-1| > |1-y^2|$. Because only the $n = 0$ term survives in (86), one is left with

$$Z_1(y) = \kappa \frac{y^{1-k}}{1-y^2}. \quad (87)$$

Similarly, on \mathcal{J}_2 (76) is easily calculated to be

$$\begin{aligned} \tilde{Z}_2(y) &= \kappa \int_{\mathcal{J}_2} \frac{(e^{\kappa u} - y^2)^k}{(e^{\kappa u} - 1)^k} du \\ &= \kappa y^{2k} \int_z^{z+1} \frac{(1 - \epsilon e^{\kappa r} y^{-2})^k}{(1 - \epsilon e^{\kappa r})^k} dr \\ &= \kappa y^{2k} \frac{\sum_{n=0}^{\infty} \binom{k}{n} (\epsilon y^{-2k})^n}{\sum_{n=0}^{\infty} \binom{k}{n} \epsilon^n} \stackrel{\epsilon \rightarrow 0}{=} \kappa y^{2k}, \quad (88) \end{aligned}$$

and finally

$$Z_2(y) = \kappa \frac{y^{1+k}}{1-y^2}. \quad (89)$$

Therefore, modulo determining a couple of coefficients, the index reads

$$\mathcal{I}(y, \zeta) = n_1(\zeta) \frac{y^{1-k}}{1-y^2} + n_2(\zeta) \frac{y^{1+k}}{1-y^2}. \quad (90)$$

As expected, there is no sign of flow time τ anywhere in the index or in the thimble integrals even though the cycles $\mathcal{J}_{1,2}$ were formed as a result of flow by steepest descent, which is a necessary condition for relative homology classes to represent a topological invariant for all flow times.

7.2. FI parameter dependency

But how do we know FI parameter dependency e.g., ζ -dependency, of intersection coefficients? We will only focus on rank-one gauge group G that applies to the present case.

Localization helps the infinite-dimensional path integral of the field theory to reduce to an integral of the 1-loop determinant over the finite-dimensional moduli space of supersymmetric configurations, \mathcal{M} which inherits the same topological structure as that of the spacetime torus T^2 for elliptic genus, which is a complex torus $\mathbb{C}/(\mathbb{Z} + \mathbb{t}\mathbb{Z})$ (modulo group automorphisms). It is of real dimension 2 or equivalently, a $1d$ complex torus. Taking the limit $\mathbb{t} \rightarrow i\infty$ casts u_i into the form (33) where the moduli space is the u -space $X = \mathbb{C}/\mathbb{Z}$ with a non-compact imaginary direction. The 1-loop determinant is basically a meromorphic top-form on \mathcal{M} , and therefore considering the poles of the $\mathcal{N} = (2,2)$ chiral and vector multiplet contribution to the full 1-loop determinant, which are present in localization locus, elliptic genus is given by

$$\mathcal{I}_{T^2} = \lim_{\epsilon \rightarrow 0} \int_{\Gamma} \frac{dD}{\kappa D} e^{\frac{-D^2}{2\epsilon^2} - \zeta D} \oint_{-\gamma} g(\mathbb{t}, z, u, D) du, \quad (91)$$

with $g(\mathbb{t}, z, u, D)$ being a holomorphic function in u expressed as the product of vector and chiral 1-loop determinants,

$$g(\tau, z, u, D) = \mathcal{Z}_V(q, y) \prod_j \mathcal{Z}_{\Psi, Q_j}(\mathbb{t}, z, u, D, \zeta), \quad (92)$$

and again u parametrizes gauge holonomy and takes values in \mathcal{M} . The poles of the function g form a subset $\mathcal{M}_{\text{sing}} \subset \mathcal{M}$. Residues of the poles $u_j^P \in \mathcal{M}_{\text{sing}}$ as instructed by JK residue operation will enter the index formula through a correct choice of integration cycle γ . However, there are also poles in the D -plane that entirely lie on the imaginary axis. It is shown in [15–17] by a relatively straightforward computation that in the D -plane, there is only a pole D_j^P for each chiral multiplet of charge Q_j that approaches the real axis as $u \rightarrow u_j^P$ at a rate of ϵ^2 where ϵ shows the radius of an infinitesimally small closed loop going around every pole in u -plane. The rest of these poles stay far away from the real axis in D -plane. Thus, picking a deformed contour Γ for the D -integral that avoids singularities of $\mathcal{M}_{\text{sing}}$ and D_j^P at the same time away from the point $D = 0$ is feasible but subject to taking the localization limit $e \rightarrow 0$ in (91) after the limit $\epsilon \rightarrow 0$ is performed. This, therefore, will necessitate a choice of some Cartan subalgebra-valued “displacement vector” $\delta \in \mathfrak{h}_{\mathbb{C}}$ where $\mathfrak{h}_{\mathbb{C}}$ is the complexified D -plane, such that

$$\Gamma : \Re(D) = \mathbb{R}, \Im(D) = \delta. \quad (93)$$

The final result of JK residue operation will not depend on this vector anyways.

Now at this stage, before integrating out D and taking the localization limit, we modify the FI term using an analytically continued version of “Higgs scaling” [17]:

$$\begin{aligned} \mathcal{I}_{T^2} &= \lim_{\epsilon \rightarrow 0} \int_{\Gamma} \frac{dD'}{\kappa D'} e^{-\frac{D'^2}{2\epsilon^2} - \frac{1}{2\epsilon^2} \zeta' D'} \oint_{\gamma} g(\mathbb{t}, z, u, D') du \\ &= \lim_{\epsilon \rightarrow 0} e^{-\frac{1}{2\epsilon^2} |\zeta'|^2} \int_{\Gamma'} \frac{dD'}{\kappa(D' + \zeta')} e^{-\frac{D'^2}{2\epsilon^2}} \\ &\quad \times \oint_{\gamma} g(\mathbb{t}, z, u, D' + \zeta') du \end{aligned}$$

where $\zeta' = \zeta \epsilon^2$ is held fixed and $D \rightarrow D'$. Here, ζ' is a pure imaginary number and thus it is assumed that $\zeta' \in \mathfrak{t}$ with \mathfrak{t} being the Lie algebra of the maximal torus of G . For brevity we will drop the primes in the following.

Now everything said in the paragraph before (93) remains equally valid except the contour deformation in $\mathfrak{h}_{\mathbb{C}}$ is done away from the point $D = -\zeta$, so

$$\Gamma' : \Re(D) = \mathbb{R}, \Im(D) = \delta - \zeta. \quad (94)$$

Taking the limit $\epsilon \rightarrow 0$ will move the very close poles D_j^P to the point $D = -\zeta$, and therefore the normalized elliptic genus (94) may be equally computed by

$$\mathcal{I}_{T^2} = \lim_{\epsilon \rightarrow 0} e^{-\frac{1}{2\epsilon^2} |\zeta|^2} \int_{\Gamma'} \frac{dD}{\kappa(D + \zeta)} e^{-\frac{D^2}{2\epsilon^2}} \oint_{\gamma} g(\mathbb{t}, z, u) du. \quad (95)$$

For a theory with a rank-one gauge group, we can simply take δ to be some small positive number so the D -integral in (94) will produce a factor

$$i\pi - i\pi \text{erf}\left(\zeta/\sqrt{2\epsilon^2}\right), \quad (96)$$

which upon taking localization limit $e \rightarrow 0$, yields

$$\kappa \Theta(\zeta), \quad (97)$$

where $\Theta(\zeta)$ is the Heaviside step function, equal to 0 if $\zeta > 0$ along imaginary direction and zero otherwise. The appearance of $\Theta(\zeta)$ is in line with the expectation that the index should depend on FI parameters in a discontinuous fashion. So taking ζ to be pure imaginary or real will not change the result of index. This suggests that analytic continuation of ζ to pure imaginary numbers, does indeed keep the index intact and therefore FI chambers will stay the same. Elliptic genus is then given as

$$\mathcal{I}_{T^2} = \Theta(\zeta) \oint_{\gamma} g(\mathbb{t}, z, u) du, \quad (98)$$

which is a special case of formula (3.62) in [15]. Translating (98) into relative homology language, one immediately finds

$$\mathcal{I}_{T^2} = \sum_i n_i(\zeta) \int_{\mathcal{J}_i} g(\mathbb{t}, z, u) du. \quad (99)$$

This equation is gist of the correspondence between Lefschetz thimbles in the present work and the JK residue operation in [15, 16]. We observe that as expected, the only dependency of the index on ζ is via the intersection coefficients $n_i(\zeta)$ not the thimble integrals.

Regarding the ζ -dependency of n_i in $2d$ theories versus $1d$, since one can introduce a theta term in $2d$ theories with a compact target space, interpolating between $\zeta > 0$ and $\zeta < 0$ is done safely so the actual index is invariant under ζ -deformation unless $\theta = 0$ [36]. Making a non-compact direction in moduli space \mathcal{M} is essentially done by considering the dimensional reduction of the elliptic genus on a circle known as “ χ_y genus”, which in our formalism results in

$$\mathcal{I} = \sum_i n_i(\zeta) \int_{\mathcal{J}_i} \lim_{\mathbb{t} \rightarrow i\infty} g(\mathbb{t}, z, u) du. \quad (100)$$

where \mathcal{J}_i are 1-manifolds in $X = \mathbb{C}^\times$. We note that first the limit is applied and then thimble integrations are carried out. χ_y genus enjoys phase transitions as n_i may jump by varying ζ .

In the case of a 2-node quiver with total gauge group $U(1)^2$, there are two domains or “phases” separated by a wall at $\zeta_2 = -\zeta_1 = 0$ in $\mathfrak{t} = \mathfrak{h}$ (Abelian) which allows us to set $\delta = \zeta_2 \equiv \zeta$ located on the imaginary line $i\mathbb{R} \subset \mathfrak{t}$. So the homology coefficients jump at $\zeta = 0$, motivating us to write

$$n_i(\zeta) := n_i \Theta(\zeta), \quad n_i \in \mathbb{Z}. \quad (101)$$

This is the simplest example of wall-crossing in $1d$ rank-one supersymmetric quantum theories.

7.3. χ_y genus and R -anomaly removal

Let us take a $U(1)$ theory with k chiral multiplets of gauge charge 1. One knows that on the torus T^2 , there are monodromies in the now D -independent function $g(\mathbb{t}, z, u) = \mathcal{Z}_V(\mathbb{t}, z) \prod_j \mathcal{Z}_{\Psi, Q_j}(\mathbb{t}, z, u)$ because the left-moving R -symmetry is anomalous. This is readily translated in the g -function as

$$\begin{aligned} g(\mathbb{t}, z, u + l_1 + l_2 \mathbb{t}) &= e^{2\pi i z l_2 k} g(\mathbb{t}, z, u) \\ &\equiv y^{2l_2 k} g(\mathbb{t}, z, u), \end{aligned} \quad (102)$$

for $l_1, l_2 \in \mathbb{Z}$. So one has to set $z \in \mathbb{Z}/k$ to have a single-valued $g(\mathbb{t}, z, u)$. The requirement means that $y^k = \pm 1$.

The solutions $x = 0e^{\kappa \Re(u)}$, $x = \infty e^{\kappa \Re(u)}$ relax the R -anomaly removal condition $y^{2k} = 1$, that is not necessarily forced along $\mathcal{J}_1, \mathcal{J}_2$. In other words, it is not immediate that a flow by steepest descent along \mathcal{J}_1 and \mathcal{J}_2 fulfills $y^{2k} = 1$. It is noticeable that the good regions G_1, G_2 are open discs of small radii ϵ centered at “saddle points” $u = z, z + 1 \in \mathfrak{F}$, respectively. Lefschetz thimbles attached to saddle points along imaginary-infinity rims $\mathfrak{B}_-, \mathfrak{B}_+$ are to flow out of and in these regions at appropriate flow times $-\infty, \tau^*$. Once the limit $\epsilon \rightarrow 0$ is taken, we have Stokes rays $ST_2 : \Re(u) = \Re(z), ST_3 : \Re(u) = \Re(z) + 1$.

Since the difference between the two lines is a shift by $1 \in \mathbb{Z}$, then both ST_2 and ST_3 point to the same element in $X = \mathbb{C}/\mathbb{Z}$ and therefore $ST_2 = ST_3$. Indeed, $\mathcal{J}_1, \mathcal{J}_2$ end on a unique Stokes ray to connect to good regions. Thus, for instance,

$$\Im(S_{\text{eff}}(-i\infty, y)) = \Im(S_{\text{eff}}(i\infty, y)) \quad (103)$$

in $\mathfrak{F} \subset X$, fixes $y^{2k} = 1$ (see eq. (78)). Extending this treatment to the whole of u -space entirely, we conclude the same result for all thimbles by the fact that gauge invariance of the Wilson loop corresponding to $U(1)_R$ gauge field - R -symmetry- calls for the condition (103) to be satisfied along all Stokes rays connected to the saddle points $z + N$ for integer N . This is how R -anomaly is globally removed along every Lefschetz thimble in $\mathbb{P}_{k-1}(\mathbb{C})$ model.

What about the saddles at $u = z + N$? The thimbles attached to these saddles are explicitly given by

$$\{x = y^{L(\tau)} \mid L : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}, \quad y^{2k} = 1\}. \quad (104)$$

which are classified into two types of \mathcal{J} -cycles and their corresponding \mathcal{K} -cycles:

$$\mathcal{J}_3 = \{u(-\infty) = z, \quad u(\tau^*) = z + 1, \quad \Im(u) = 0, \quad \Re(u(\tau)) = \frac{z\tilde{L}(\tau)}{2}\}, \quad (105)$$

$$\mathcal{K}_3 = \{u(-\infty) = 0, \quad u(\tau^*) = 1, \quad \Im(u) = 0\}, \quad (106)$$

$$\tilde{\mathcal{J}}_3 = \{u(-\infty) = z, \quad u(\tau^*) = z + 1, \quad \Im(u) = 0, \quad \Re(u(\tau)) = \frac{z\tilde{L}(\tau)}{2}\}, \quad (107)$$

$$\tilde{\mathcal{K}}_3 = \{u(-\infty) = 0, \quad u(\tau^*) = 1, \quad \Im(u) = 0\}, \quad (108)$$

where all the cycles are restricted to the fundamental domain. Real functions L, \tilde{L} determined by flow equations are smooth in τ and should entail asymptotic properties $L(-\infty) = \tilde{L}(-\infty) = 2, L(\tau^*) = \tilde{L}(\tau^*) = 2 + 2/z$. Since $u = z + N$ for integer N are saddle points, the lines connecting them naively seem to be Stokes rays. However, two obstructions exist for this to come true. First, for $\Im(z) = 0$, the Stokes rays would hit the poles already on the real line. For $\Im(z) \neq 0$, the Stokes rays can be defined vacuously only when $\mathcal{J}_3 \cong \tilde{\mathcal{J}}_3$. In this case, one can still calculate the index in a very simple way over the contractible loop $\gamma_c := \mathcal{J}_3 - \tilde{\mathcal{J}}_3 \cong \{z\}$ where the homology classes of $\tilde{\mathcal{J}}_3$ and $\tilde{\mathcal{J}}_3$ only differ by an exact 1-form. Otherwise, the homology classes of $\tilde{\mathcal{J}}_3$ and $\tilde{\mathcal{J}}_3$ being not path homotopic, have to be related to each other by a singular 1-form $\mathcal{Z}^{1L} du$, whose integral over the non-contractible loop $\gamma_d := \mathcal{J}_3 - \tilde{\mathcal{J}}_3$ yields a non-vanishing residue, provided that imaginary shift of z would be of very small order of $\delta \ll \epsilon$ (see fig. [14]). This amounts to JK residue, thus giving the index of $\mathbb{P}_{k-1}(\mathbb{C})$ correctly. There is no well-defined notion of Stokes ray in this case.

Nonetheless, none of these solutions really matter for us because their contribution to path integral vanish via $\mathcal{Z}^{1L}(y, u = z + N) = 0$ for integer N unless $z = 0$. So the type of factor we see in this equation is essentially tied to the degeneracy of saddle points in the current problem as already seen from the exactness of saddle point approximation that we would need to regularize the 1-loop determinant to get a hold of a vanishing Hessian \mathcal{H} at those saddle points. In other words, for $k \mp i\delta$ -bifundamentals in a linear Abelian quiver with rank-one gauge groups, we find that

$$\mathcal{H}(u_j^*) = \pm i\delta(-1)^{-k} \quad j = 1, 2. \quad (109)$$

The point $\delta = 0$ is a supersymmetric point. Assume we are away from this point and we now have the formula

$$\begin{aligned} \mathcal{I} &= \int_{\mathcal{J}_3} \mathcal{Z}^{1L}(x, y) \frac{dx}{x} - \int_{\tilde{\mathcal{J}}_3} \mathcal{Z}^{1L}(x, y) \frac{dx}{x} \\ &\equiv \kappa^{-1} \text{Res}_{u=0} (\mathcal{Z}^{1L}(e^{\kappa u}, y)) \end{aligned} \quad (110)$$

instead. So the value of integral (110) localizes at point $u = 0$, taking into account that now $z \in \mathbb{Z}/(k \mp i\delta)$ for

small positive δ upon forcing the R -anomaly condition $y^{2k \pm 2i\delta} = 1$. In this case we find

$$\begin{aligned}
\mathcal{I} &= \pi y^{1-k \pm i\delta} (1-y^2)^{k \mp i\delta - 1} (y^2-1)^{-k \pm i\delta} (1-y^{2k \mp 2i\delta}) \\
&\quad \times \csc(k \mp i\delta) \pi \\
&= \frac{\pi}{\sin(k \mp i\delta) \pi} \frac{y^{1-k \pm i\delta} - y^{1+k \mp i\delta}}{1-y^2} \\
&= \frac{\pm i}{\delta} (-1)^k \frac{y^{1-k \pm i\delta} - y^{1+k \mp i\delta}}{1-y^2} \\
&= \frac{\pm i}{\delta} (-1)^k y^{-k+1} \sum_{i=0}^{k-1} y^{2i} \quad \text{for } \delta \rightarrow 0,
\end{aligned} \tag{111}$$

which modulo the factor $\frac{\pm i}{\delta} \delta (-1)^k$, is exactly equal to the index formula (39b). The appearance of the divergent factor is no coincidence and a good intuition comes from multi-instanton calculus, in two ways:

- 1 - The T -character calculation of the refined index of $\mathcal{N} = 4$ quiver quantum mechanics introduces an integral over the center of mass motion of dyonic bound-state, that needs to be renormalized as shown in [37].
- 2 - This is actually analogous to the divergent integral over zero mode direction in the holomorphized path space of $\mathcal{N} = 2$ supersymmetric quantum mechanics [9]. This direction is, indeed, a “center of mass” coordinate of instanton-anti-instanton correlated event that can, for instance, be regularized via Faddeev-Popov method.

Of course we avoided the “infinite” factor $\frac{\pm i}{\delta} \delta (-1)^k$ throughout this paper by moding out the path homotopic flow lines in the main formula due to degeneracy of saddle points in \mathfrak{B}_{\pm} .

8. ON-THE-WALL INDEX: GENERALIZED XYZ MODEL

The main lesson we have learned so far is that the integration cycle formula

$$\gamma = \sum_{i=1}^J n_i(\zeta) \mathcal{J}_i \tag{112}$$

where J shows the cardinality of \mathbf{J} -set, is in some sense the equation of the wall itself which was first noticed in [17]. It is on a Stokes ray that consists of a combination of \mathcal{J} -cycles that renders the refined index *on the wall*. If the index takes only an identical non-vanishing function of y in some chamber(s) and zero in the remaining chambers, as in all examples covered in section [4], on-the-wall index will be that of the index in those non-vanishing chambers. For example, in the 2-node quiver system expounded at length in the present paper, if the gauge group of quiver quantum mechanics is $G = U(1)^2$, the wall is associated

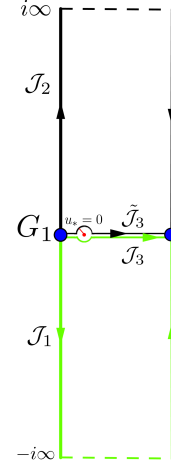


Figure 14. Deformed γ integration cycle, γ_d is a cycle that is formed in the following way. Observe that the integral along imaginary directions, on both \mathcal{J} -cycles cancel out and the only non-vanishing piece left is the one along “flat” directions, $\mathcal{J}_3, \tilde{\mathcal{J}}_3$. So Picard-Lefschetz theory now gives $\gamma_d = n_1 \mathcal{J}_3 + n_2 \tilde{\mathcal{J}}_3$. Since there is a pole at $u_* = 0$, we have given a shift of $\mp i\delta$ to k to make thimbles $\mathcal{J}_3, \tilde{\mathcal{J}}_3$ dodge the pole. This is done as $z \in \mathbb{Z}/(k \mp i\delta) \sim \mathbb{Z}/k \pm i\delta$ to remove R -anomaly, analytically dislocating real saddles to above and below real line. At the end, the index would not depend on this small parameter as the limit $\delta \rightarrow 0$ is taken.

with the condition $\zeta_2 = -\zeta_1 = 0$ that is a point in ζ -space, and the chamber $\zeta_2 > 0$ is where the index is non-vanishing (check the explanation under fig. [7]).

As a closing section, we want to briefly weigh in on the most curious example of a closed quiver that is a generalization of XYZ model in subsection [6.1] with a cubic superpotential [20] that will put this idea into perspective. Since the index is non-zero in the two FI chambers, studying this case can shed more light on the interpretation of γ and what it means to make sense out of an ill-defined homology cycle.

The quiver is made out of a total of $2p + 2$ chiral multiplets with field content of $[\mathbf{X}, \mathbf{Y}, \mathbf{Z}]$ in the bifundamental representation of the gauge group for $p \geq 1$:

$$\mathbf{X} = (X_1, X_2) \quad \mathbf{Y} = (Y_1, \dots, Y_p) \quad \mathbf{Z} = (Z_1, \dots, Z_p). \tag{113}$$

A cubic superpotential asks for $R_{\mathbf{X}} + R_{\mathbf{Y}} + R_{\mathbf{Z}} = 2$ where R_i stands for the R -charge of each field and all chiral fields in the same multiplet carry the same charge assigned to them. The 1-loop determinant is given as

$$\mathcal{Z}^{1L} = \frac{\kappa^2 y^{4+2p} (1 - x_1 y^{R_X - 2})^2 \left(\frac{x_2 - y^{R_Z - 2}}{x_2 - y^{R_Z}} \frac{x_1 - x_2 y^{R_Y - 2}}{x_1 - x_2 y^{R_Y}} \right)^p}{(1 - y^2)^2 (1 - x_1 y^{R_Z})^2} \tag{114}$$

The integration cycle is

$$\gamma = \sum_{i=1}^4 n_i(\zeta) \mathcal{J}_i \equiv \mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 + \mathcal{J}_4 \tag{115}$$

Therefore, calculating the thimble integrals, one finds

$$\begin{aligned} Z_1(y) &= \lim_{\substack{x_1 \rightarrow 0 \\ x_2 \rightarrow 0}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^{4-2p}}{(1-y^2)^2}, \\ Z_2(y) &= \lim_{\substack{x_1 \rightarrow \infty \\ x_2 \rightarrow \infty}} \mathcal{Z}^{1L} = \kappa^2 \frac{1}{(1-y^2)^2}, \\ Z_3(y) &= \lim_{\substack{x_1 \rightarrow 0 \\ x_2 \rightarrow \infty}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^4}{(1-y^2)^2}, \\ Z_4(y) &= \lim_{\substack{x_1 \rightarrow \infty \\ x_2 \rightarrow 0}} \mathcal{Z}^{1L} = \kappa^2 \frac{y^{2p}}{(1-y^2)^2}, \end{aligned}$$

and the on-the-wall index is

$$\begin{aligned} \mathcal{I} &= \frac{y^{-2p} (y^4 - y^{2p} - y^{4+2p} + y^{4p})}{(1-y^2)^2} \\ &= \sum_{j=1}^p (j-1) \left(y^{-2(p-j)} + y^{2(p-j)} \right) - p. \end{aligned} \quad (116)$$

Comparing this result to that of [20], we find that, as expected, it is made out of the superposition of BPS configurations of the quiver theory in two chambers sharing the same wall γ . The Stokes jumps are thus computed by

$$\begin{aligned} \mathcal{I} &= \frac{y^{-2p} (y^4 - y^{2p} - y^{2p+4} + y^{4p})}{(1-y^2)^2} + p - p \\ &= \frac{y^{4-2p} + (p-1) + (p-1)y^4 + y^{2p}}{(1-y^2)^2} - \frac{2py^2}{(1-y^2)^2} \\ &\quad - \left(\frac{p+py^4}{(1-y^2)^2} - \frac{2py^2}{(1-y^2)^2} \right) \end{aligned}$$

The reason for the appearance of self-dual term can be explained in two senses. It is perhaps an artifact of adding a regularization term in the effective action that is not fully invariant under the action of G . Or, complexifying the compact Lie group G can indeed lead to an integration cycle $\tilde{\mathcal{C}}$ constructed, off the Stokes rays, in the form of

$$\tilde{\mathcal{C}} = \sum_{i,j} n_{ij} \tilde{\mathcal{J}}_i \times \tilde{\mathcal{J}}_j, \quad (117)$$

since in reality, complexification of G is isomorphic to $G \times G$ and the fully complexified theory should then have $G \times G$ -symmetry [6]. Here, we have defined the $\tilde{\mathcal{J}}$ -cycles in such a way that the thimble integrals are rescaled according to $\tilde{Z}_i^2 = (y - y^{-1})^2 Z_i$ which defines the $G \times G$ -invariant index to be

$$\mathcal{I}(\zeta) = \frac{\kappa^{-2}}{(y - y^{-1})^2} \sum_{i,j} n_{ij}(\zeta) \tilde{Z}_i \times \tilde{Z}_j. \quad (118)$$

Here, $[\tilde{Z}_i, \tilde{Z}_j] = [\tilde{\mathcal{J}}_i, \tilde{\mathcal{J}}_j] = 0$ for all i, j , and the overall factor is just the vector multiplet contribution which is $G \times G$ -invariant as well. Therefore, it is only in this

latter sense that in two chambers sharing the wall γ with non-vanishing values of index, the integration cycles are

$$\begin{aligned} \tilde{\mathcal{C}}_{0e^{i\theta_1}} &= \tilde{\mathcal{J}}_1^2 + (p-1) \tilde{\mathcal{J}}_2^2 \\ &\quad + (p-1) \tilde{\mathcal{J}}_3^2 + \tilde{\mathcal{J}}_4^2 + 2p \tilde{\mathcal{J}}_2 \times \tilde{\mathcal{J}}_3, \\ \tilde{\mathcal{C}}_{0e^{i\theta_2}} &= p \tilde{\mathcal{J}}_2^2 + p \tilde{\mathcal{J}}_3^2 + 2p \tilde{\mathcal{J}}_2 \times \tilde{\mathcal{J}}_3, \end{aligned}$$

where in light of the prescription (48) we have defined

$$\gamma = \tilde{n}_1(\zeta) \tilde{\mathcal{C}}_{0e^{i\theta_1}} + \tilde{n}_2(\zeta) \tilde{\mathcal{C}}_{0e^{i\theta_2}} + 0 \tilde{\mathcal{C}}_{0e^{i\theta_3}}, \quad (119)$$

with \tilde{n}_i being in general a function of $\Theta(\zeta)$ in the i th chamber of ζ -space. There are a total of three chambers and when $\theta = \theta_3$, the contribution of $\tilde{\mathcal{C}}_{0e^{i\theta_3}}$ will drop out of γ . The jumping of the coefficients n_{ij} in transitioning from chamber 1 to 2 is then given by

$$n_{11} \rightarrow n_{11} - n_{44} + n_{22}, \quad n_{44} \rightarrow n_{44} - n_{11} + n_{33}, \quad (120)$$

and the remaining coefficients will not jump.

9. KNOTS AND LEFSCHETZ THIMBLES

A possible interesting feature of supersymmetric theories is their relation to knot homologies and more recently Khovanov homology [24, 38–43]. In math literature, it is known from a classic work by Ozsvath and Szabo [44] that any null-homologous knot K in S^3 (or in general any closed, oriented three-manifold) can be given a Floer-homology description where, roughly speaking, homology cycles are flow lines of the Morse function over the three-manifold. Knot invariants in the form of a categorification of Alexander polynomials are then obtained by calculating the knot Floer homology groups of the three-manifold by applying different surgeries along the knot such as crossing resolutions and connected sums. Floer homology is an infinite-dimensional analog of finite-dimensional Morse homology, which is best suitable for studying invariants of 3d quantum field theories [45].

A knot is a circle embedded in a three dimensional space. Higher dimensions give a lot of freedom to deform knots. On the other hand, a codimension 1 knot turns out to be very restricting. However, taking knots in a 3d space and projecting them onto a 2d space seems to provide a crucial link between Lefschetz thimbles and the flow dynamics of quiver quantum mechanics in the following sense.

Let G be a gauge group that is either a unitary group or product of them. The dimensional reduction of elliptic genus on S^1 produces a moduli space for all supersymmetric configurations to which the path integral localizes. Call this moduli space Σ and observe that it is of complex dimension $\alpha = \text{rank}(G)$. To be specific, suppose for instance, Σ is the u -space of an $\mathcal{N} = 4$ quiver theory with total gauge group $G = U(1)^\alpha$ where l th $U(1)$ factor is decoupled. Then Σ has certain singularities that periodically happen so we will instead take $\Sigma^F \subset \Sigma$, the fundamental

domain in Σ . Take an alternating mirror symmetric knot K in \mathbb{R}^3 such that $\pi_1(\mathbb{R}^3 \setminus K) \cong \mathbb{Z}$. Suppose the first relative homology $H_1(\Sigma^F, \Sigma_{\tau^*}^F; \mathbb{Z})$ is defined and \mathcal{J}_i are 1-cycles with homologies $H_1(\Sigma^F, \Sigma_{\tau^*}^F; \mathbb{Z})$. Again, $\Sigma_{\tau^*}^F$ is a good region to be reached at $\tau > \tau^*$ by flow lines $\mathcal{J}_i \subset \Sigma^F$. We define the effective action $S_{\text{eff}} = -\ln(\mathcal{Z}^{1L})$ in which $\mathcal{Z}^{1L}(x_1, \dots, \hat{l}, \dots, x_\alpha)$ corresponds to the 1-loop determinant of the gauged quiver quantum theory with l th node removed. The complex Morse function is $h_{\mathbb{C}} = S_{\text{eff}}^{\text{reg}}$ with the usual regularization in effect.

Identify good regions in Σ^F by the crossings of K and downward flow lines of the equations

$$\frac{d\bar{u}^j}{d\tau} = -g^{i\bar{j}} \frac{\partial h_{\mathbb{C}}}{\partial u^i}, \quad \frac{du^i}{d\tau} = -g^{i\bar{j}} \frac{\partial \bar{h}_{\mathbb{C}}}{\partial \bar{u}^j} \quad (121)$$

($g^{i\bar{j}}$ is a suitable metric on u -space), by the line segments connecting these crossings. A Lefschetz thimble contributing to the knot invariant always connects two good regions which correspond to either $+-$ or $-+$ crossings for the connecting segment. Then we have defined a projection of the knot K onto the space Σ^F , called \mathcal{PK} . (See fig. [15] to check what projected trefoil knot looks like in this setup). Then, inspired by (61) the knot \mathcal{PK} is conjectured to be given by

$$\mathcal{PK} = \sum_{i=1}^{2^{\alpha-1}} (-1)^{t_i-1} \mathcal{J}_i. \quad (122)$$

where there are $2^{\alpha-1}$ distinct types of saddle rims in Σ^F .

The HOMFLY polynomial corresponding to an unknot is given as

$$f(\lambda, y)(\text{unknot}) = (\lambda^{-1} - \lambda)(y^{-1} - y)^{-1}. \quad (123)$$

This leads to the $\mathfrak{sl}(k)$ knot invariant of the unknot if we put $\lambda = y^k$. But note that this exactly corresponds to the refined Witten index of the Abelian 2-node quiver $\mathcal{N} = 4 \mathbb{P}_{k-1}(\mathbb{C})$ model- discussed at full length in section [5] and [7] where unknot is topologically equivalent to the superposition of two Lefschetz thimbles:

$$\text{unknot} = \mathcal{J}_1 - \mathcal{J}_2. \quad (124)$$

So one may conjecture that the invariant (61) may detect the alternating mirror symmetric knots upon the understanding of Reidemeister moves and crossing resolutions. The best candidate is Alexander polynomial $\Delta_K(t)$

that can be drawn from HOMFLY polynomial. It is a knot invariant that could be obtained from (122) once y is identified with an appropriate function of t due to the symmetry $\Delta_K(t) = \Delta_K(t^{-1})$. In a way, this sort of visualization of knots suggests what geometric sense we can draw from e.g. $\Delta_K(t)$ by relating every term in it directly to a specific cycle in the moduli space of a quantum system with a Hamiltonian that is conserved along every one of these cycles.

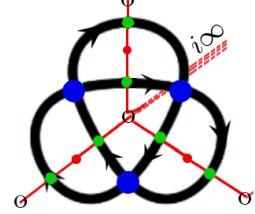


Figure 15. A possible planar thimble decomposition for trefoil knot in a regularized gauged quantum mechanics \mathcal{A} . Good regions are shown to be at the crossings and \mathcal{J} cycles are the directed curves and red crosses are singularities in the moduli space of some quiver theory. Red lines symbolize \mathcal{K} cycles that intersect the good cycles orthogonally and flow to infinity along imaginary direction, consistent with the structure of thimbles in fig. [7]. The green circles represent the saddle points. Therefore, $\mathcal{PK} = \sum_{i=1}^6 (-1)^{t_i-1} \mathcal{J}_i$, such that $\mathcal{J}_i \in H_1(\Sigma^F, \Sigma_{\tau^*}^F)$ for Σ being the u -space of \mathcal{A} . The knot formula (122) will not obviously detect trefoil knot so we need to consider a different gauge group G or a complexification of G .

ACKNOWLEDGMENT

We would like to thank Philip Argyres, Francesco Benini, Gerald Dunne, Babak Haghighat, Chris J. Howls, Atsushi Kanazawa, Tatsuhiro Misumi, Goncalo Oliveira, Radmila Sazdanovic, Thomas Schäfer, Shamil Shakirov, Shu-Heng Shao, Mithat Ünsal and Wenbin Yan for useful discussions. We are so indebted to Can Kozcaz for proposing this project and Yuya Tanizaki for his collaboration in the early stages of this work. We want to thank Shing-Tung Yau finally for his warm hospitality in Department of Mathematics and the Center for Mathematical Sciences and Applications (CMSA) at Harvard University, where most of current work was completed in the Fall of 2015. This work was supported by the DOE grant DE-SC0013036.

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